



13 $\dot{x} = f(x)$; $f(a) = 0 \rightarrow x = a$ is eq. point

$\lim_{t \rightarrow \infty} x(t) = a$ for all x starting } $x = a$ is a
around a } +ve attractor

negative attractor ($t \rightarrow \infty$ is $t \rightarrow -\infty$)



$$\dot{x} = Ax$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$P = \text{trace}(A)$$

$$q = \det(A)$$

§2.4 1st integrals

$$\dot{x} + x = 0 \quad \text{mult by } \dot{x}$$

$$\hookrightarrow \dot{x} \dot{x} + x \dot{x} = 0$$

$$\left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 \right)' = 0$$

$$\hookrightarrow \frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 \right) = 0$$

$$\hookrightarrow \frac{1}{2} \overset{E_k}{\dot{x}^2} + \frac{1}{2} \overset{E_p}{x^2} = \text{constant}$$

$$\hookrightarrow F(x, \dot{x}) = \frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 \Rightarrow \frac{dF}{dt} = 0$$

1st integral

$$\begin{cases} F_1(x, \dot{x}) = C_1 \\ F_2(x, \dot{x}) = C_2 \\ F_3(x, \dot{x}, t) = C_3 \end{cases} \quad \left. \begin{aligned} x &= h_1 \cos t + h_2 \sin t \\ \textcircled{1} \quad x^2 + \dot{x}^2 &= h_1^2 + h_2^2 \\ \textcircled{2} \quad x \cos t - \sin t \dot{x} &= h_1 \end{aligned} \right\}$$

In phase space:

circle
a manifold



$$F = \frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2 = C$$

Def 1 Orbital derivative:

$$F: \mathbb{R}^n \rightarrow \mathbb{R}$$

$$x: \mathbb{R} \rightarrow \mathbb{R}^n$$

Orbital derivative:

time indep

$$F(x_1, \dots, x_n), t$$

$$x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$L_t F = \frac{d}{dt} F(x_i(t))$$

$$= \sum_{i=1}^n \frac{\partial F}{\partial x_i} \frac{dx_i}{dt} + \frac{\partial F}{\partial t}$$

Def 1 $\dot{x} = f(x)$

time indep.

$f(x)$ is called a 1st integral of $\dot{x} = f(x)$ if

$$L_t F = 0$$

example

$$\ddot{x} + x - \frac{1}{2}x^2 = 0$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + \frac{1}{2}x^2 \end{cases} \quad \text{1 eq. pts} \quad \begin{cases} y = 0 \\ -x + \frac{1}{2}x^2 = 0 \end{cases}$$

$$\text{1 eq. pts } (x, y) = (0, 0), (2, 0)$$

nonlin.

Spring

$\downarrow x$

linearized

center point

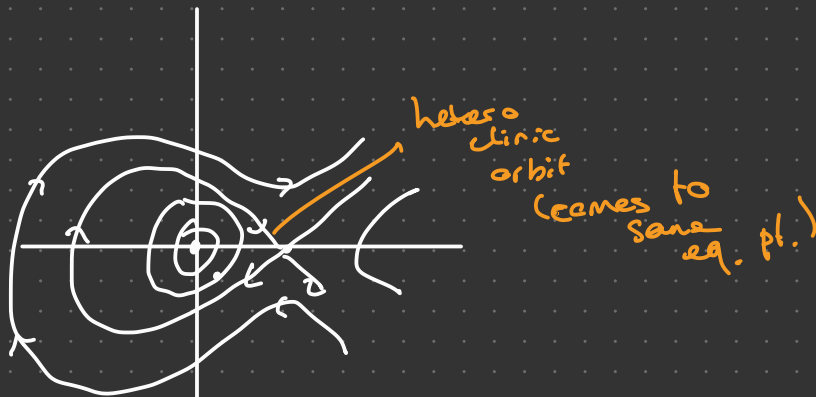
$$\lambda = \pm i$$

no conclusion
for the N.L.D.E

saddle point

(chap. 7)

NLDE also
a saddle point



$$\ddot{x} + x - \frac{1}{2}x^2 = 0 \quad \cdot \dot{x}$$

$$\hookrightarrow \dot{x} \ddot{x} + x \dot{x} - \frac{1}{2}x^2 \dot{x} = 0 \Rightarrow \frac{d}{dt} \left(\frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 - \frac{1}{6} x^3 \right) = 0$$

$$\hookrightarrow \int_t \sim \frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 - \frac{1}{6} x^3 = \text{constant}$$



For the saddle point:

$$\frac{1}{2} \dot{x} + \frac{1}{2} x^2 - \frac{1}{6} x^3 = \frac{1}{2} \cdot 0 + \frac{1}{2} - \frac{8}{6} = \frac{2}{3}$$

$\hookrightarrow$ divide out known root $x=2 \rightarrow$ quadr. eq. or use Cardano

$\hookrightarrow$ gives another root + homoclinic orbit

HERE BC OF
SYMMETRY

example Hamiltonian Systems:

$$\begin{cases} \dot{p}_i = - \frac{\partial H}{\partial q_i} \\ \dot{q}_i = \frac{\partial H}{\partial p_i} \end{cases}$$

$$H = H(p_1, \dots, p_n, q_1, \dots, q_n)$$
$$L_H H = \frac{d}{dt} H = \sum_n \left(\underbrace{\frac{\partial H}{\partial p_n}}_{-\frac{\partial H}{\partial q_n}} \dot{p}_n + \underbrace{\frac{\partial H}{\partial q_n}}_{\frac{\partial H}{\partial p_n}} \dot{q}_n \right) = 0$$

Def 1 $F: \mathbb{R}^n \rightarrow \mathbb{R}$; $F \in C^\infty$

$$\frac{\partial F}{\partial x} \Big|_a = 0 \Leftrightarrow \left(\frac{\partial F}{\partial x_i} \right) \Big|_a = \vec{0}, \quad \vec{a} = \begin{pmatrix} a_1 \\ \vdots \end{pmatrix}$$

$\hookrightarrow$ vector

$x=a$ is a critical point. The point $x=a$ is called a non degenerate critical point of the function $F(x)$ if the determinant

$$\left| \frac{\partial^2 F}{\partial x^2}(a) \right| \neq 0$$

$\hookrightarrow$ matrix

$$\underline{n=1}: F(x) = F(a) + \frac{\partial F}{\partial x}(a)(x-a) + \frac{1}{2} \frac{\partial^2 F}{\partial x^2}(a)(x-a)^2 + \dots$$

$$\underline{n=2}: F(x_1, x_2) = F(a_1, a_2) + \frac{\partial F}{\partial x_1}(a)(x_1 - a_1) + \frac{\partial F}{\partial x_2}(a)(x_2 - a_2) + \frac{1}{2} \frac{\partial^2 F}{\partial x_1^2}(a)(x_1 - a_1)^2 + \frac{\partial^2 F}{\partial x_1 \partial x_2}(a)(x_1 - a_1)(x_2 - a_2) + \frac{1}{2} \frac{\partial^2 F}{\partial x_2^2}(a)(x_2 - a_2)^2 + \dots$$

$$F(x) = F(a) + \nabla F(a) \cdot (x-a) + \frac{1}{2} (x_1-a_1 \quad x_2-a_2) \begin{pmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2^2} \end{pmatrix} \begin{pmatrix} x_1-a_1 \\ x_2-a_2 \end{pmatrix} + \dots$$

matrix vector

$$F(\hat{x}) = F(\hat{a}) + \nabla F(\hat{a}) \cdot (\vec{x} - \vec{a}) + \frac{1}{2} (\vec{x} - \vec{a})^T \frac{\partial^2 F}{\partial x^2}(\hat{a}) (\vec{x} - \vec{a}) + \dots$$

Def 1 If $x=a$ is a nondegenerate critical point of the C^∞ function $F(x)$, then $F(x)$ is called a Morse function in the neighborhood of $x=a$

Theorem 1 $x=0$ is a nondegenerate critical point.

Assume: $F(x) = F_0 - C_1 x_1^2 - C_2 x_2^2 - \dots - C_n x_n^2 + C_{n+1} x_{n+1}^2 + \dots + C_k x_k^2 + \dots$
 $C_i > 0$

Then: in a neighborhood of $x=0$ there exists a diffeomorphism (that is a transformation $x \mapsto y$ which is one-to-one unique, C^1 inverse exists) which transforms $F(x)$ to the form:

$$G(y) = G(0) - y_1^2 - y_2^2 - \dots - y_n^2 + y_{n+1}^2 + \dots + y_k^2$$

example

$$\ddot{x} + x + x^3 = 0$$

first integral: $F(x, \dot{x}) = \frac{1}{2} \dot{x}^2 + \frac{1}{2} x^2 + \frac{1}{4} x^4 = \text{const.}$

eq. pts on crit. pt:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - x^3 \end{cases}$$

$$\begin{cases} F(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{4}x^4 \\ \frac{\partial F}{\partial x} = x + x^3 \\ \frac{\partial F}{\partial y} = y \end{cases}$$

$$\Rightarrow y = 0$$

$$x(1+x^2) = 0 \quad \text{neglect complex solutions} \Rightarrow x = 0$$

$$F(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 \quad \leadsto \quad G(y_1, y_2) = y_1^2 + y_2^2$$

$$\begin{cases} y_1 = x_2 \\ y_2 = x_1 \sqrt{1 + \frac{1}{4}x_1^2} \end{cases}$$

Previous example

Now we know locally closed orbits near $(0,0)$ = center pt.

orbit inside cent: ① go to ∞ **cross**

② go to eq. **no eq. pt.**

③ spiral **symmetry**



Mixed terms are not an issue $\rightarrow$ coord. transf.

$$\hookrightarrow x^2 + xy + y^2 = c$$

$$\hookrightarrow (x + \frac{1}{2}y)^2 - \frac{1}{4}y^2 + y^2 = \text{const.}$$

$$\hookrightarrow \tilde{x}^2 + \frac{3}{4}y^2 = \text{const.}$$

Evolution of a volume element

$$\vec{x} = f(x)$$

$$D(t) = g^t D(0)$$

$$g^t(x): \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$v(t) = \int_{D(t)} dx = \int_{D(0)} \left| \frac{\partial g^t(x)}{\partial x} \right| dx$$

where is point x after t

Jacobian

$t=0, D(0), v(0)$

$D(t), v(t)$



THEOREM: $\frac{dv}{dt} \Big|_{t=0} = \int_{D(0)} \nabla \cdot f dx$

PROOF

$$g^t(x) = x + \underbrace{x}_f t + O(t^2) \text{ for } t \rightarrow 0$$

$$\frac{\partial g^t(x)}{\partial x} = I + \frac{\partial f}{\partial x} t + O(t^2)$$

$$\left| \frac{\partial g^t(x)}{\partial x} \right| = \begin{bmatrix} 1 + t \frac{\partial f_1}{\partial x_1} + O(t^2) & t \frac{\partial f_1}{\partial x_2} + O(t^2) & \dots \\ t \frac{\partial f_2}{\partial x_1} + O(t^2) & 1 + t \frac{\partial f_2}{\partial x_2} + O(t^2) & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$= 1 + t \sum \frac{\partial f_i}{\partial x_i} + O(t^2)$$

$$v(t) = \int_{D(0)} \left[1 + t \sum \frac{\partial f_i}{\partial x_i} + O(t^2) \right] dx \text{ (for } t \rightarrow 0)$$

$$\frac{dv}{dt} \Big|_{t=0} = \int_{D(0)} \sum \frac{\partial f_i}{\partial x_i} dx = \int_{D(0)} \nabla \cdot f dx$$

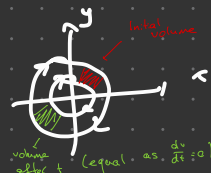
Example

$$\dot{x} = -x \quad ; \quad \frac{dv}{dt} \Big|_{t=0} = \int_{D(0)} \nabla \cdot f \, dx = \int_{D(0)} \frac{d}{dx} (-x) \, dx = \int_{D(0)} -1 \, dx = -V(0)$$

Example

$$\begin{cases} \ddot{x} + x = 0 \\ \dot{x} = y \\ \dot{y} = -x \end{cases}$$

$$\frac{dv}{dt} \Big|_{t=0} = \int_{D(0)} \nabla \cdot f \, dx = \int_{D(0)} \left(\frac{d}{dx} \left(\frac{y}{-x} \right) \right) (-x) \, dx = 0$$



$$\dot{x} = f(x) = \begin{pmatrix} y \\ -x \end{pmatrix}$$

CH.3

→ $\dot{x} = f(x) \rightarrow$ critical points, stability

Linear Equations: $\dot{y} = Ay$ with $\det A \neq 0$

↳ This means: $Ay = 0 \rightarrow y = A^{-1}0 = 0$

↳ for eq. points.

↳ Unique equilibrium points

↳ solve: $\xi e^{\lambda t}$, λ are eigenvalues

$$L: Z = T^{-1} A T Z$$

$$y = T Z \quad \text{↳ Jordan form}$$

§3.1: $n=2$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$$



§3.2: $n=3$, Cardano

§3.3: $\dot{x} = f(x)$, nonlinear.

$x=0$ is a critical point

Taylor: $\dot{x} = \underbrace{Ax}_{\substack{\text{matrix} \\ n \times n}} + \underbrace{g(x)}_{\substack{\text{nonlinear} \\ \text{vector function} \\ \text{km} \rightarrow \text{kg} \\ \text{kg} \cdot \text{m} \rightarrow \text{kg} \cdot \text{m} \\ \text{kg} \cdot \text{m} \cdot \text{s}^{-2} \rightarrow \text{kg} \cdot \text{m} \cdot \text{s}^{-2}}}$; $\det A \neq 0$

THEOREM: If $x=0$ is a ^{negative} positive attractor for the linearized equation, then $x=0$ is also a ^{negative} positive attractor for the nonlinear equation

THEOREM: If the matrix A has an eigenvalue with positive real part, then the critical point $x=0$ is not a positive attractor for the nonlinear equation

Blow-up Techniques

$$\begin{cases} \frac{dx}{dt} = 2xy \\ \frac{dy}{dt} = x^2 + y^2 \end{cases} \Rightarrow \text{only 1 eq. point: } (0,0)$$

$$\nabla f = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \begin{pmatrix} 2xy \\ x^2 + y^2 \end{pmatrix} = uy \rightarrow \text{inconclusive!}$$

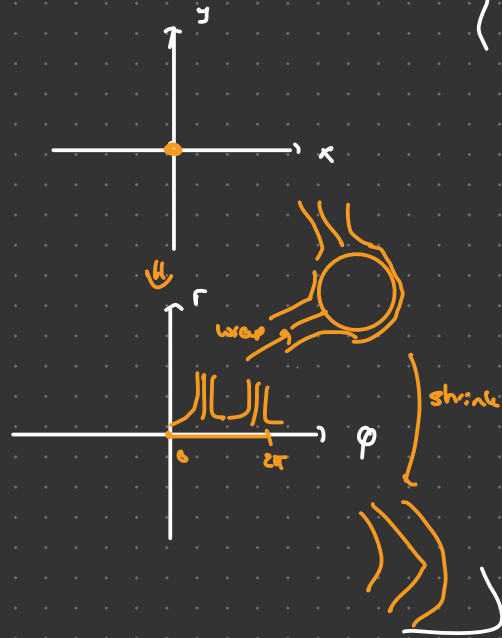
$$\begin{pmatrix} f_{1,x} & f_{1,y} \\ f_{2,x} & f_{2,y} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$$

Critical point is degenerate

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \Rightarrow \frac{1}{2} \frac{dy^2}{dx} = \frac{x^2}{2x} + \frac{y^2}{2x}$$

Introduce polar coordinates

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \end{cases}$$



$$\begin{cases} \frac{dx}{dt} = \frac{dr}{dt} \cos \varphi - r \sin \varphi \frac{d\varphi}{dt} = 2r^2 \cos \varphi \sin \varphi \quad \cdot \cos \varphi \quad \cdot -\sin \varphi \\ \frac{dy}{dt} = \frac{dr}{dt} \sin \varphi + r \cos \varphi \frac{d\varphi}{dt} = r^2 \quad \cdot \sin \varphi \quad \cdot \cos \varphi \end{cases}$$

$$\begin{cases} \textcircled{1}: \frac{dr}{dt} \\ \textcircled{2}: r \frac{d\varphi}{dt} \end{cases} \begin{aligned} &= 2r^2 \cos^2 \varphi \sin \varphi + r^2 \sin \varphi \\ &= -2r^2 \cos \varphi \sin^2 \varphi + r^2 \cos \varphi \end{aligned}$$

$$\begin{cases} \frac{dr}{dt} = 2r^2 \cos^2 \varphi \sin \varphi + r^2 \sin \varphi \\ \frac{d\varphi}{dt} = -2r \cos \varphi \sin^2 \varphi + r \cos \varphi \end{cases}$$

we want to get rid of the r^2 in dr/dt .

$$\hookrightarrow \begin{cases} r(t) = \tilde{r}(s) \\ \varphi(t) = \tilde{\varphi}(s) \end{cases} \quad \text{with } \frac{ds}{dt} = r \quad \text{time transformation}$$

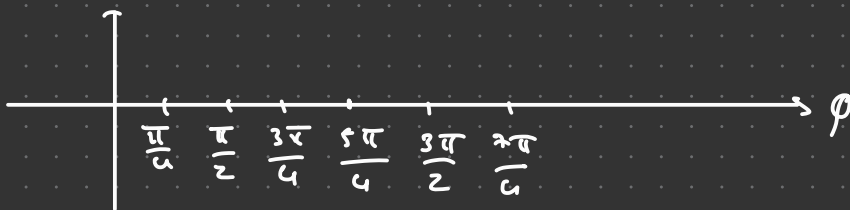
$$\hookrightarrow \frac{dr}{dt} = \frac{d\tilde{r}}{ds} \cdot \frac{ds}{dt} \quad \text{other method, divide 2 eq's.}$$

$$\hookrightarrow \begin{cases} \frac{d\tilde{r}}{ds} = 2r \cos^2 \varphi \sin \varphi + r \sin \varphi \\ \frac{d\tilde{\varphi}}{ds} = -2 \cos \varphi \sin^2 \varphi + \cos \varphi \end{cases}$$

eq. points at $r=0$:

$$-2 \cos \varphi \sin^2 \varphi + \cos \varphi = 0$$

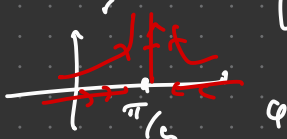
$$\hookrightarrow \varphi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$$



linearize system around:

$$(1, \varphi) = (0, \pi/4): \begin{cases} \dot{r} = r\sqrt{2} \\ (\varphi - \frac{\pi}{4})' = -\sqrt{2} (\varphi - \frac{\pi}{4}) \end{cases}$$

Saddle point



Do the same at other eq. points,

$$u \quad (r, \varphi) = (0, \frac{\pi}{2}) ; \quad \dot{r} = r$$

$$(\varphi - \frac{\pi}{2})' = \varphi - \frac{\pi}{2}$$

→ unstable
structure
unknown

↳ Blow up:

$$\begin{cases} r = \rho + \rho \cos \theta \\ \varphi = \frac{\pi}{2} + \rho \sin \theta \end{cases}$$

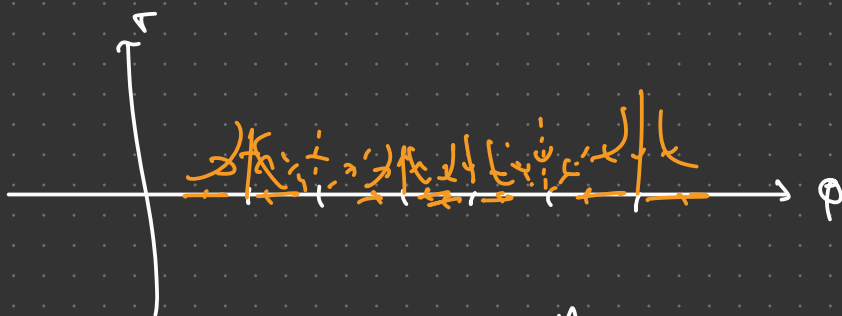
$$u \quad (r, \varphi) = (0, \frac{3\pi}{4}) ; \quad \dot{r} = \sqrt{2} r$$

$$(\varphi - \frac{3\pi}{4})' = -\sqrt{2} (\varphi - \frac{3\pi}{4})$$

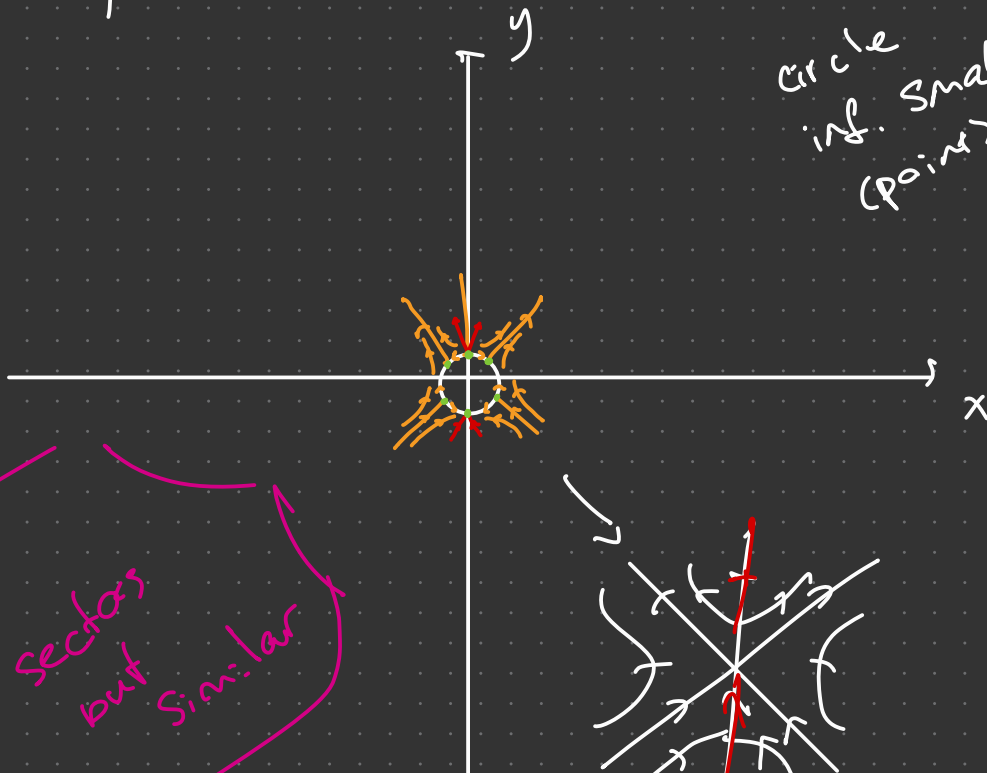
saddle
point

others:

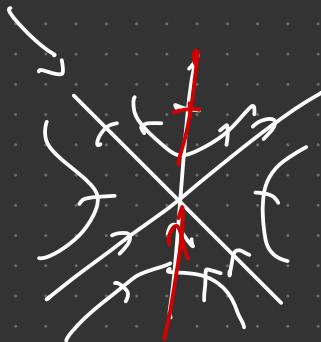
$$\begin{aligned} (r, \varphi) = (0, \frac{\pi}{4}) : & \begin{cases} \dot{r} = -\sqrt{2} r \\ (\varphi - \frac{5\pi}{4})' = +\sqrt{2} (\varphi - \frac{5\pi}{4}) \end{cases} \\ (r, \varphi) = (0, \frac{3\pi}{2}) : & \begin{cases} \dot{r} = -r \\ (\varphi - \frac{3\pi}{2})' = -(\varphi - \frac{3\pi}{2}) \end{cases} \\ (r, \varphi) = (0, \frac{7\pi}{4}) : & \begin{cases} \dot{r} = -\sqrt{2} r \\ (\varphi - \frac{7\pi}{4})' = \sqrt{2} (\varphi - \frac{7\pi}{4}) \end{cases} \end{aligned}$$



circle
inf. small
(points)



TTE:
6 sectors
but similar



6.5 Multiple Time-Scales Perturbation Method

Not in Book

Practical: 11/7 or 11/13 afternoon

Books: - Kevlahan & Cole
- Nayfeh
- Murdock

$$\ddot{x} + x + \varepsilon \dot{x} = 0; \quad 0 < \varepsilon \ll 1$$

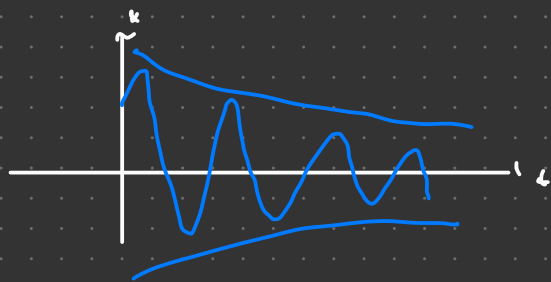
$$x(t) \sim x_0(t) + \varepsilon x_1(t) + \dots; \quad x_i(t) \text{ and its derivatives are } O(1)$$

$$\text{ICs: } x(0) = a_0 + \varepsilon a_1 + \dots$$

$$\dot{x}(0) = b_0 + \varepsilon b_1 + \dots$$

$$O(1): \ddot{x}_0 + x_0 = 0 \quad \Rightarrow x_0(t) = a_0 \cos t + b_0 \sin t$$

$$O(\varepsilon): \ddot{x}_1 + x_1 = -\dot{x}_0 \quad \Rightarrow x_1(t) = a_1 \cos t + b_1 \sin t - \frac{1}{2}(b_0 + a_0 t) \cos t - \frac{1}{2} b_0 t \sin t \quad \left. \begin{array}{l} \text{secular} \\ \text{terms} \end{array} \right\}$$



fast oscillation, slow damping
 t εt

$$x(t) = \tilde{x}(t, \frac{\varepsilon t}{\varepsilon}) \quad \text{or: } t_0 = t, t_1 = \varepsilon t, \dots \quad x(t) = \tilde{x}(t_0, t_1, \dots)$$

$$x(t) = \tilde{x}(t, \tau), \quad \tau = \varepsilon t$$

$$\frac{dx}{dt} = \frac{d\tilde{x}}{dt} = \frac{\partial \tilde{x}}{\partial t} + \frac{\partial \tilde{x}}{\partial \tau} \frac{\partial \tau}{\partial t} = \tilde{x}_t + \varepsilon \tilde{x}_\tau$$

$$\frac{d^2 \tilde{x}}{d\tau^2} = \frac{d^2 \tilde{x}}{d\tau^2} = \frac{d}{dt} \left(\frac{d\tilde{x}}{d\tau} \right) = \frac{d}{dt} (\tilde{x}_t + \varepsilon \tilde{x}_\tau) = \tilde{x}_{tt} + 2\varepsilon \tilde{x}_{t\tau} + \varepsilon^2 \tilde{x}_{\tau\tau}$$

So, our ODE becomes the following PDE:

$$\tilde{x}_{tt} + 2\varepsilon \tilde{x}_{t\tau} + \varepsilon^2 \tilde{x}_{\tau\tau} - \tilde{x} + \varepsilon (\tilde{x}_t + \varepsilon \tilde{x}_\tau) = 0$$

$$\tilde{x}(t, \tau) \sim x_0(t, \tau) + \varepsilon x_1(t, \tau) + \dots; \quad x_i \text{ are } O(1) \text{ for } t = O\left(\frac{1}{\varepsilon}\right)$$

Substituting expansion into PDE:

$$O(1): \quad x_{0,tt} + x_0 = 0 \quad \Rightarrow \quad x_0(t, \tau) = A_0(\tau) \cos t + B_0(\tau) \sin t$$

$$O(\varepsilon): \quad x_{1,tt} + x_1 = -2x_{0,t\tau} - x_{0,t}$$

$$\begin{aligned} x_{1,tt} + x_1 &= -2 \left(-\frac{dA_0}{d\tau} \sin t + \frac{dB_0}{d\tau} \cos t \right) - (-A_0 \sin t + B_0 \cos t) \\ &= \left(2 \frac{dA_0}{d\tau} + A_0 \right) \sin t - \left(2 \frac{dB_0}{d\tau} + B_0 \right) \cos t \end{aligned}$$

To avoid secular terms (i.e. unbounded terms in x_1 , like $t \sin t$, $t \cos t$) we have to set the coefficients of $\sin t$ and $\cos t$ in the right-hand side of the eq. for x_1 equal to 0

↳ No solutions in the RHS of the hom. eq. or no resonant functions / excitations

$$\Rightarrow \begin{cases} \frac{dA_0}{d\tau} + A_0 = 0 \\ 2 \frac{dB_0}{d\tau} + B_0 = 0 \end{cases} \Rightarrow \begin{cases} A_0(\tau) = A_0(0) e^{-\tau/2} \\ B_0(\tau) = B_0(0) e^{-\tau/2} \end{cases}$$

$$\begin{aligned} x_{\text{appr}}(t) &= x_0(t, \tau) + \varepsilon x_1(t, \tau) \\ &= A_0(0) e^{-\tau/2} \cos t + B_0(0) e^{-\tau/2} \sin t + \varepsilon x_1(t, \tau) \end{aligned}$$

↳ Bounded function for $t = O\left(\frac{1}{\varepsilon}\right)$

$$\ddot{x}_{\text{appr}} + x_{\text{appr}} + \varepsilon \dot{x}_{\text{appr}} = \dots + \dots = \varepsilon^2 R(t, \varepsilon)$$

↳ Previously: $R = O(t)$
Now: $R = O(1)$

Repeat the proof:

$$|x(t) - x_{\text{appr}}(t)| = O(\varepsilon) \text{ for } t = O\left(\frac{1}{\varepsilon}\right)$$

$$|x(t) - x_0(t, \tau)| \leq |x(t) - (x_0 + \varepsilon x_1) + \varepsilon x_1| \leq \underbrace{|x(t) - x_{\text{appr}}(t)|}_{O(\varepsilon)} + \underbrace{|\varepsilon x_1|}_{O(\varepsilon) \text{ for } t = O\left(\frac{1}{\varepsilon}\right)} \\ \leq O(\varepsilon) \text{ for } t = O\left(\frac{1}{\varepsilon}\right)$$

Example 1 Van der Pol

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}$$

$$x(t) = \tilde{x}(t, \tau) \sim x_0(t, \tau) + \varepsilon x_1(t, \tau) + \dots$$

Goal: construct an $O(\varepsilon)$ accurate approximation of $x(t)$ for $t = O\left(\frac{1}{\varepsilon}\right)$

ODE becomes PDE:

$$\tilde{x}_{tt} + 2\varepsilon \tilde{x}_{t\tau} + \varepsilon^2 \tilde{x}_{\tau\tau} + \tilde{x} = \varepsilon(1 - \tilde{x}^2)(\tilde{x}_t + \varepsilon \tilde{x}_\tau)$$

Substitute $x_0 + \varepsilon x_1 + \dots$ for $\tilde{x}$ into the PDE and collect terms of $O(1), O(\varepsilon), \dots$

$$O(1): x_{0,tt} + x_0 = 0$$

$$\cos^3 = \frac{3}{4}\cos + \frac{1}{4}\cos 3$$

$$O(\varepsilon): x_{1,tt} + x_1 = -2x_{0,t\tau} + (1 - x_0^2)x_{0,t}$$

$$\hookrightarrow O(1): x_0(t, \tau) = A_0(\tau)\cos t + B_0(\tau)\sin t$$

$$O(\varepsilon): x_{1,tt} + x_1 = -2\left(-\frac{dA_0}{d\tau}\sin t + \frac{dB_0}{d\tau}\cos t\right)$$

$$+ (1 - (A_0\cos t + B_0\sin t)^2)(-A_0\sin t + B_0\cos t)$$

$$= \left(-2\frac{dB_0}{d\tau} + B_0 - \frac{1}{4}B_0(A_0^2 + B_0^2)\right)\cos t$$

$$+ \left(2\frac{dA_0}{d\tau} - A_0 + \frac{1}{4}A_0(A_0^2 + B_0^2)\right)\sin t$$

$$+ (\dots)\cos(3t) + (\dots)\sin(3t)$$

To avoid secular terms, A_0 and B_0 have to satisfy:

$$\begin{cases} \frac{1}{2} \frac{dA_0}{d\tau} - A_0 + \frac{1}{4} A_0 (A_0^2 + B_0^2) = 0 \\ -\frac{1}{2} \frac{dB_0}{d\tau} + B_0 - \frac{1}{4} B_0 (A_0^2 + B_0^2) = 0 \end{cases}$$

$$\hookrightarrow \begin{cases} \frac{dA_0}{d\tau} = \frac{1}{2} A_0 (1 - \frac{1}{4} (A_0^2 + B_0^2)) \\ \frac{dB_0}{d\tau} = \frac{1}{2} B_0 (1 - \frac{1}{4} (A_0^2 + B_0^2)) \end{cases}$$

$$x_0(t, \tau) = A_0(\tau) \cos t + B_0(\tau) \sin t$$

$$|x(t) - x_0(t, \tau)| = O(\epsilon) \text{ for } t = O(\frac{1}{\epsilon})$$

$$\begin{cases} A_0(\tau) = r(\tau) \cdot \cos(\varphi(\tau)) \\ B_0(\tau) = r(\tau) \sin(\varphi(\tau)) \end{cases} \quad \left. \vphantom{\begin{cases} A_0(\tau) = r(\tau) \cdot \cos(\varphi(\tau)) \\ B_0(\tau) = r(\tau) \sin(\varphi(\tau)) \end{cases}} \right\} \text{Introduce Polar coordinates}$$

$$\begin{cases} \frac{dA_0}{d\tau} = \frac{dr}{d\tau} \cos(\varphi(\tau)) - r(\tau) \sin(\varphi(\tau)) \frac{d\varphi}{d\tau} = \frac{1}{2} r \cos \varphi (1 - \frac{1}{4} r^2) \\ \frac{dB_0}{d\tau} = \frac{dr}{d\tau} \sin \varphi + r \cos \varphi \frac{d\varphi}{d\tau} = \frac{1}{2} r \sin \varphi (1 - \frac{1}{4} r^2) \end{cases}$$

$\begin{aligned} &\cdot \cos \varphi \\ &\cdot -\sin \varphi \\ &\cdot \sin \varphi \\ &\cdot \cos \varphi \end{aligned}$

$$\begin{aligned} \textcircled{1} & \begin{cases} \frac{dr}{d\tau} = \frac{1}{2} r (1 - \frac{1}{4} r^2) \\ r \frac{d\varphi}{d\tau} = 0 \Rightarrow \frac{d\varphi}{d\tau} = 0 \quad (r > 0) \Rightarrow \varphi(\tau) = \varphi(0) \end{cases} \\ \textcircled{2} & \end{aligned}$$

$r \rightarrow r = 2 \text{ stable eq. pt.}$

$$x_0(t, \tau) = A_0 \cos t + B_0 \sin t$$

$$= r \cos \varphi \cos t + r \sin \varphi \sin t$$

$$= r \cos(t - \varphi(0))$$

$$r \frac{dr}{d\tau} = \frac{1}{2} r^2 (1 - \frac{1}{4} r^2)$$

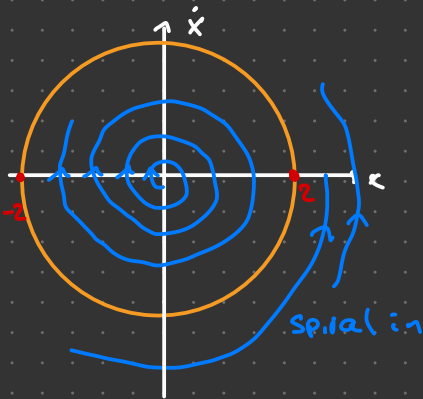
$$\frac{1}{2} \frac{d^2 r}{d\tau^2} = \frac{1}{2} r^2 (1 - \frac{1}{4} r^2)$$

$$r^2 = R$$

$$\frac{dR}{d\tau} = R(1 - \frac{1}{2}R)$$

$$R(1 - \frac{1}{2}R) \frac{dR}{d\tau} = 1 \Rightarrow \left(\frac{A}{R} + \frac{B}{1 - \frac{1}{2}R} \right) \frac{dR}{d\tau} = 1$$

$$\hookrightarrow r(\tau) = \sqrt{\frac{ur^2(0)e^\tau}{u-r^2(0)r^2(0)e^\tau}}$$



$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}$$

$$\dot{x} = y$$

$$\dot{y} = -x + \varepsilon(1 - x^2)y$$

O: periodic solution (limit cycles)

6 Periodic Solutions for $\dot{x} = f(x)$

4.1 Bendixon's Criterion

$$\text{In } \mathbb{R}^2; \begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \quad ; \quad D \subset \mathbb{R}^2$$

simply connected
no holes
 f, g are C^1 in D

Theorem 4.1 System (*) can only have periodic solutions in D if $\nabla \cdot \begin{pmatrix} f \\ g \end{pmatrix} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ changes sign in D or if $\nabla \cdot \begin{pmatrix} f \\ g \end{pmatrix} = 0$ in D

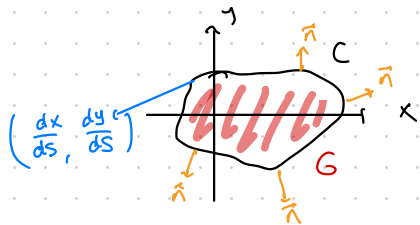
for sure there are no periodic solutions in D when $\nabla \cdot \begin{pmatrix} f \\ g \end{pmatrix}$ is sign definite in D

Proof Suppose there is closed curve C in D . G is the interior of C

Closed curve C corresponds with a periodic sol.

$$\text{Gauss' div: } \iint_G \nabla \cdot \begin{pmatrix} f \\ g \end{pmatrix} dV = \int_C \begin{pmatrix} f \\ g \end{pmatrix} \cdot \vec{n} ds$$
$$\vec{n} = \pm \begin{pmatrix} dy/ds \\ -dx/ds \end{pmatrix}$$

$$\int_C \begin{pmatrix} f \\ g \end{pmatrix} \cdot \vec{n} ds = \pm \int_C \begin{pmatrix} f \\ g \end{pmatrix} \cdot \begin{pmatrix} g \\ -f \end{pmatrix} ds = 0$$



Example $\ddot{x} + p(x)\dot{x} + q(x) = 0$
 $p(x) > 0$ for all x

$$\begin{cases} \dot{x} = y \\ \dot{y} = -q(x) - p(x)y \end{cases}$$

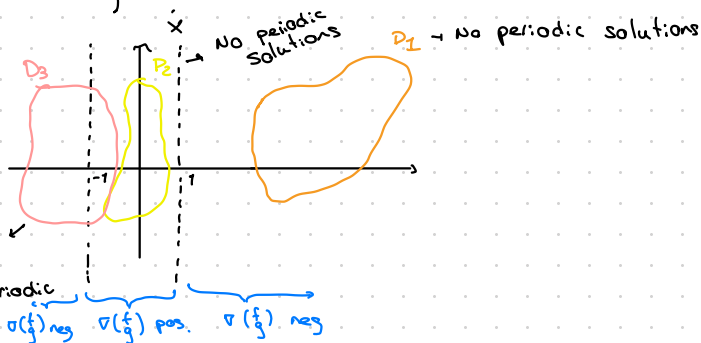
$$\nabla \cdot \begin{pmatrix} f \\ g \end{pmatrix} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 - p(x) = -p(x) < 0$$

↳ Sign definite divergence $\Rightarrow$ No periodic solutions in D

Example $\ddot{x} + x = \varepsilon(1 - x^2)x$
Van der Pol, $\varepsilon > 0$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + \varepsilon(1-x^2)y \end{cases}$$

$$\nabla \left(\frac{f}{g} \right) = 0 + \varepsilon(1-x^2) = \varepsilon(1-x^2)$$



In $\mathbb{R}^n$ with $n \geq 3$ the Bendixson criterion is not valid

$$\begin{cases} \dot{x} = f(x, y, z) = -y & \rightarrow x = c_1 \cos t + c_2 \sin t \\ \dot{y} = g(x, y, z) = x & \rightarrow y = c_1 \sin t - c_2 \cos t \\ \dot{z} = h(x, y, z) = z & \rightarrow z = c_3 e^t \end{cases}$$

$\nabla \left(\frac{f}{g} \right) = f_x + g_y + h_z$ is sign definite and there are periodic solutions

§4.2 Geometric Definitions

$$\dot{x} = f(x), \quad x \in D \subset \mathbb{R}^n, \quad t \in \mathbb{R}$$

an invariant set M is a set in $\mathbb{R}^n$ such that if $x(0)$ is contained in M then $x(t)$ is contained in M for $t \in \mathbb{R}$

solution $x(t)$ with $\forall x(0) = x_0 \xrightarrow{\text{phase space}} \text{orbit } \gamma(x_0)$

If at $t = t_1$: $x = x(t_1)$, then $\gamma(x_0) = \gamma(x_1)$ same orbit

$$\left. \begin{array}{l} \text{for } t \geq 0: f^+(x_0) \\ \text{for } t \leq 0: f^-(x_0) \end{array} \right\} f(x_0) = f^+(x_0) \cup f^-(x_0)$$

Periodic solution: $f^-(x_0) = f^+(x_0)$

Definition

p is a ^{negative} positive limit point of the orbit $f(x_0)$

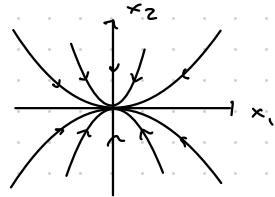
If there exists an increasing sequence of times $t_1, t_2, \dots \rightarrow \infty$ such that $x(t_1), \dots \rightarrow -p$

Example

u.u

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -2x_2 \end{cases}$$

$p = (0, 0)$ positive limit point



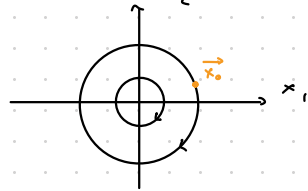
Example

u.s

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$$

What are the limit points of $f(x_0)$?

$t_n = 2\pi n + \frac{1}{n^2} + \alpha$, yielding all points on the circle



The ω -limit set of f is the set of all positive limit points of an orbit f

The α -limit set of f is the set of all negative limit points of an orbit f

Example u.u $\omega(f) = \text{origin}$

u.s $\omega(f) = f = \alpha(f)$

Theorem The sets $\alpha(f)$ and $\omega(f)$ are closed and invariant

If f^+ is bounded, then the ω -limit set is compact (bounded and closed), connected, and not empty

If $x(t; x_0)$ corresponds to $f^+(x_0)$ for $t \geq 0$, then the distance $d(x(t; x_0), \omega(f)) \rightarrow 0$ for $t \rightarrow \infty$

§4.3 The Poincaré-Bendixon's Theorem

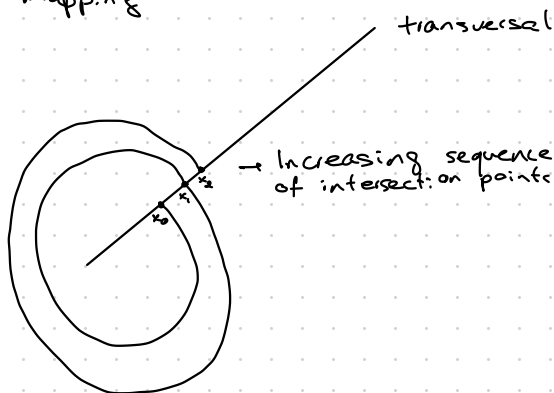
$$\dot{x} = f(x), \quad x \in \mathbb{R}^2, \quad f: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f \in C^1$$

Theorem For a positive y^+ which is bounded, but which does not correspond to a periodic solution, the ω -limit set contains a critical point or it consists of a closed orbit

Definition Points which are not critical points of $\dot{x} = f(x)$ are called ordinary points.

Transversal: smooth curve containing only ordinary points and is nowhere tangent to an orbit of $\dot{x} = f(x)$ in the phase-space

Poincaré mapping



The sequence of intersection points is increasing, but bounded above

Theorem Poincaré - Bendixson:

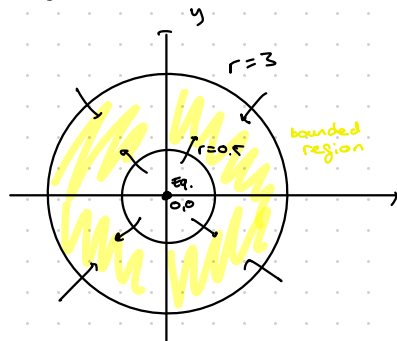
Consider $\dot{x} = f(x)$ in $\mathbb{R}^2$ and let y^+ be a bounded positive orbit. Assume that $\omega(y^+)$ only contains ordinary points. Then $\omega(y^+)$ is a periodic orbit and if $\omega(y^+) \neq y^+$, the periodic orbit is called a limit cycle

Example

$$\begin{cases} \dot{y} = y + x - y(x^2 + y^2) \\ \dot{x} = x - y - x(x^2 + y^2) \end{cases}$$

In polar coordinates:

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = 1 \end{cases}$$



Starting in $\bullet$ with a solution: for $t \rightarrow \infty$ the solution should go to a periodic solution

If $\dot{r} = -\dot{r} \uparrow$, the periodic solution is unstable

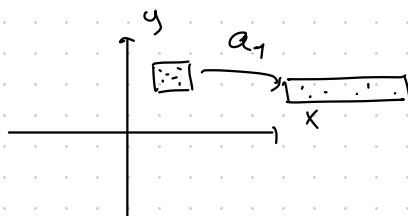
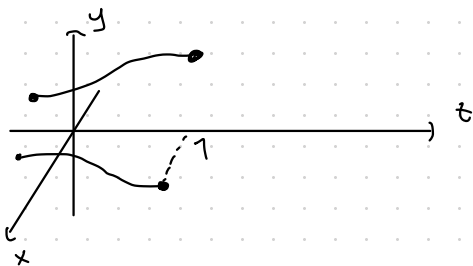
Example
4.9

Liénard equation $\ddot{x} + f(x)\dot{x} + g(x) = 0$

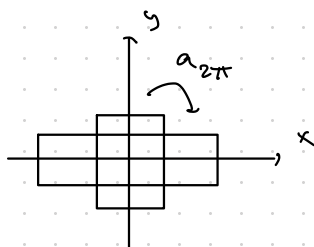
§1.5 Periodic Solutions

Example $\begin{cases} \dot{x} = 2x + \sin t \\ \dot{y} = -y \end{cases} \quad \begin{cases} \dot{x} = 2x + z \\ \dot{y} = -y \\ \dot{z} = \omega \\ \dot{\omega} = -z \end{cases}$

$$\begin{cases} x(t) = (x_0 + \frac{1}{2})e^{2t} - \frac{2}{5}\sin t - \frac{1}{5}\cos t \\ y(t) = y_0 e^{-t} \end{cases}$$



map a_1 : look "1 second" later



Does the problem have a periodic solution?

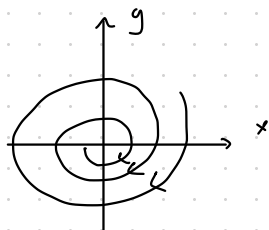
$$\begin{cases} x_0 = -\frac{1}{5} \\ y_0 = 0 \end{cases} \Rightarrow \text{periodic}$$

Chapter 5: Stability

Example $\ddot{x} + \mu \dot{x} + x = 0, \mu > 0$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - \mu y \end{cases}$$

"stay in neighborhood of equilibrium point and finally go back to it ($t \rightarrow \infty$)"



Example

$$\ddot{x} + \sin x = 0$$

periodic solutions and stability

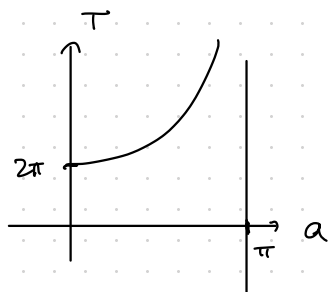
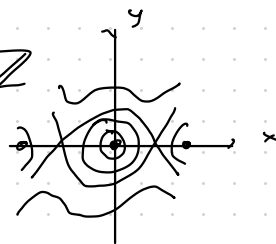
$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x \end{cases}$$

$$\begin{aligned} y &= 0 \\ x &= k\pi, k \in \mathbb{Z} \end{aligned}$$

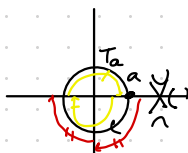
First integral:

$$\dot{x} \dot{x} + x \sin(x) = 0$$

$$\hookrightarrow F(x, \dot{x}) = \frac{1}{2} \dot{x}^2 - \cos x = C$$



$$\text{ICS: } \begin{aligned} x(0) &= a ; 0 < a < \pi \\ \dot{x}(0) &= 0 \end{aligned}$$



$$\frac{1}{2} \dot{x}^2 - \cos x = \frac{1}{2} \cdot 0^2 - \cos a$$

$$\hookrightarrow \dot{x}^2 = 2(\cos x - \cos a)$$

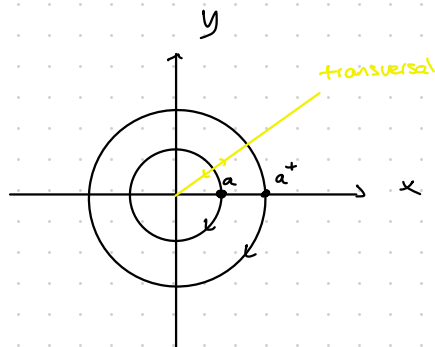
$$\dot{x} = \pm \sqrt{2} \cdot \sqrt{\cos x - \cos a}$$

$$\frac{\frac{dx}{dt}}{\sqrt{\cos x - \cos a}} = \pm \sqrt{2}$$

$$\int_0^{T_a} \dots dt$$

$$\int_0^{T_a} \frac{\frac{dx}{dt} \cdot dt}{\sqrt{\cos x - \cos a}} = \sqrt{2} T_a$$

$$\hookrightarrow \int_0^a \frac{dx}{\sqrt{\cos x - \cos a}} = \sqrt{2} T_a$$



4 orbitally stable solutions

§5.2 stability of equilibrium points

$$\dot{x} = f(t, x)$$

$$x \in \mathbb{R}^n, t \in \mathbb{R}$$

$x=0$ is a critical point

Definition stability (in the sense of Lyapunov):

$\forall \varepsilon > 0 \exists \delta(\varepsilon, t_0) > 0$ such that if $\|x_0\| < \delta$
then $\|x(t; t_0, x_0)\| < \varepsilon$ for $t > t_0$

Definition Asymptotically stable:

$x = x_0 = 0$ is (Lyapunov) stable and $\lim_{t \rightarrow \infty} \|x(t; t_0, x_0)\| = 0$

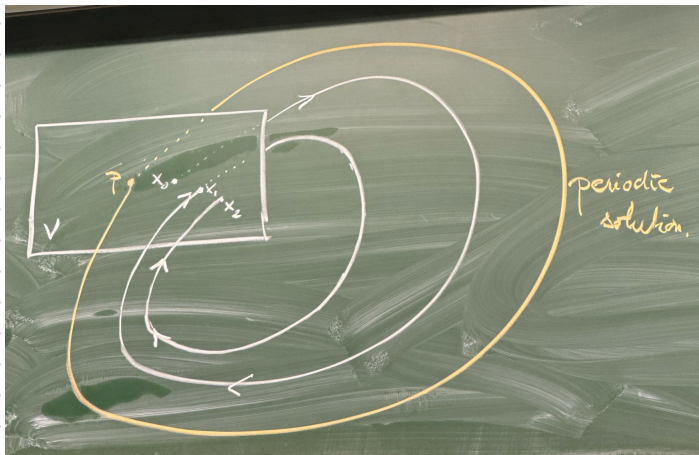
§5.3 stability of Periodic Solutions

$$\dot{x} = f(x)$$

$$x \in \mathbb{R}^n$$

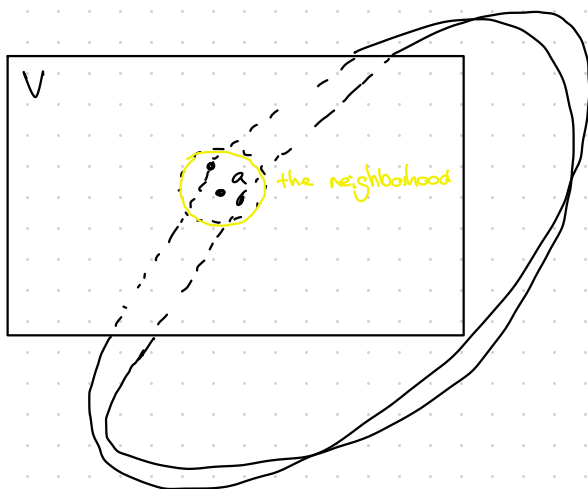
P is a Poincaré map

V is a transversal ($(n-1)$ dimensional)



Definition

The periodic solution of $\dot{x} = f(x)$ is stable (or orbitally stable) if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\|x_0 - a\| \leq \delta$, $x_0 \in V$ implies $\|P^n(x_0) - a\| \leq \varepsilon$ for $n = 1, 2, \dots$



Definition

(asymptotically stable): orbitally stable and

$$\lim_{n \rightarrow \infty} P^n(x_0) = a$$

$\hookrightarrow T$ -periodic in t

For nonautonomous systems $\dot{x} = f(t, x) \rightarrow n$ -dimensional

$$\begin{cases} \dot{x} = f(\theta, x) & \rightarrow (n+1)\text{-dimensional} \rightarrow \text{autonomous} \\ \dot{\theta} = 1; \theta(0) = 0 \pmod{T} \end{cases}$$

$$\text{with } (\theta, x) \in \underline{S^1} \times \mathbb{R}^n$$

$\hookrightarrow \text{circle } \mathbb{R} \pmod{T}$

A periodic solution (θ, x) of the system corresponds with a periodic solution of $\dot{x} = f(t, x)$

§5.4 Linearization

i) $x = a$ is a critical point of $\dot{x} = f(t, x) \Rightarrow \dot{y} = \frac{\partial f(t, a)}{\partial x} y$

lim. around $x=a$
 $\hookrightarrow$ matrix

ii) $\varphi(t)$ is a periodic solution of $\dot{x} = f(t, x)$

$$\hookrightarrow x = \varphi(t) + y(t) \Rightarrow \dot{\varphi} + \dot{y} = f(t, \varphi + y) = f(t, \varphi) + \frac{\partial f}{\partial x} y + \text{HOT}$$

$\hookrightarrow x$ small

$$\hookrightarrow \text{Since } \dot{\varphi} = f(t, \varphi) \Rightarrow \dot{y} = \frac{\partial f}{\partial x} y + \text{HOT}$$

linearize at 0

$$\dot{y} = \frac{\partial f(t, \varphi(t))}{\partial x} y$$

iii) Linearization of the mapping P around the fixed point a : (a = P(a))

$$\left. \begin{aligned} \vec{x}_{n+1} &= \vec{P}(\vec{x}_n) \\ x_n &= a + y_n \end{aligned} \right\} \begin{aligned} a + y_{n+1} &= P(a + y_n) \\ &= P(a) + \frac{\partial P(a)}{\partial x} y_n + \text{HOT} \\ \hookrightarrow y_{n+1} &= \frac{\partial P(a)}{\partial x} y_n + \text{HOT} \\ \hookrightarrow y_{n+1} &= \frac{\partial P(a)}{\partial x} y_n \end{aligned}$$

The last problem is of the form:

$$\Gamma \quad \vec{x} = A\vec{x} \quad \neg$$

$$z_{n+1} = \underbrace{A}_{\text{constant matrix}} z_n; \quad z_n = \vec{\xi} \cdot \lambda^n$$

$$x(t) = \vec{\xi} e^{\lambda t}$$

$$\neg \text{as. st. : } \operatorname{Re}(\lambda) < 0 \neq \lambda \neg$$

$$\Gamma \text{ dimensional: } \begin{aligned} z_{n+1} &= a z_n \\ z_0 &= b \end{aligned} \quad \neg$$

$$\neg \hookrightarrow z_1 = ab; \quad z_2 = a^2 b; \dots \rightarrow \text{as. st. } |a| < 1 \neg$$

$$\xi \lambda^{n+1} = A \xi \lambda^n \Rightarrow A \xi = \lambda \xi$$

LECTURE 8 Chapter 6 - Stability $\dot{x} = Ax + B(t)x$

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§6.1

$$\dot{x} = Ax$$

constant matrix

Theorem i) If all $\lambda_k, k=1, \dots, n$ satisfy $\operatorname{Re}(\lambda_k) < 0$

Then $x=0$ is asymptotically stable

If $\Phi(t)$ is the fundamental matrix, then $\|\Phi(t)\| \leq C \cdot e^{-\mu t}$ for some $\mu > 0$

ii) If $\|\Phi(t)\| \leq C$, then $x=0$ is stable

If all eigenvalues λ_k satisfy $\operatorname{Re}(\lambda_k) \leq 0$ and the dimension of the eigenspace equals the multiplicity of the λ_k for which $\operatorname{Re}(\lambda_k) = 0$

then $x=0$ is stable

iii) If there exists a λ_k with $\operatorname{Re}(\lambda_k) > 0$, $x=0$ is unstable

$$\dot{x} = Ax$$

$$\begin{aligned} \vec{x}(t) &= \Phi(t) \cdot \vec{c} \\ \vec{x}(t_0) &= \vec{x}_0 \end{aligned} \quad \left. \begin{aligned} \vec{x}_0 &= \Phi(t_0) \cdot \vec{c} \\ \vec{c} &= \Phi^{-1}(t_0) \vec{x}_0 \end{aligned} \right\}$$

$$\vec{x}(t) = \Phi(t) \Phi^{-1}(t_0) \vec{x}_0 = \Psi(t) \vec{x}_0; \Psi(t_0) = I$$

$$\text{Let } \vec{x}(t) = \Phi(t) \vec{x}_0 \text{ with } \Phi(t_0) = I$$

$$\dot{\vec{x}} = \Phi(t) \vec{x}_0 = A \Phi(t) \vec{x}_0$$

$$\Phi(t) = \begin{pmatrix} x^{(1)}(t) & \dots & x^{(n)}(t) \end{pmatrix}$$

Taylor series expansion around $t_0 = 0$:

$$\vec{x}(t) = \underbrace{\vec{x}(0)}_{\Phi(0) \vec{x}(0)} + t \underbrace{\dot{\vec{x}}(0)}_{A \Phi(0) \vec{x}(0)} + \frac{1}{2} t^2 \underbrace{\ddot{\vec{x}}(0)}_{A^2 \Phi(0) \vec{x}(0)} + \dots$$

$$\dot{x} + x = 0$$

$$\Phi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$\dot{\Phi}(t) = A \Phi(t)$$

$$= \Phi(0) \vec{x}(0) + t A \Phi(0) \vec{x}(0) + \frac{1}{2} t^2 A^2 \Phi(0) \vec{x}(0) + \dots$$

$$= \left(I + t A + \frac{1}{2} t^2 A^2 + \dots \right) \Phi(0) \vec{x}(0)$$

$$= e^{tA} x_0$$

$$\frac{d\Phi^{-1}(t)}{dt} = -\Phi^{-1}(t) A$$

$$\frac{d}{dt} (\Phi(t) \Phi^{-1}(t)) = [0]$$

$$\frac{d\Phi}{dt} \cdot \Phi^{-1} + \Phi \cdot \frac{d\Phi^{-1}}{dt} = [0]$$

$$A\Phi \cdot \Phi^{-1} + \dots$$

$$\Phi \cdot \frac{d\Phi^{-1}}{dt} = -A$$

$$\frac{d\Phi^{-1}}{dt} = -\Phi^{-1} A$$

§6.2

$$\dot{x} = A x + B(t) x$$

$\underbrace{A}_{\text{constant}} \quad \underbrace{B(t)}_{\lim_{t \rightarrow \infty} \|B(t)\| = 0}$

Example

$$\ddot{x} - \frac{2}{t} \dot{x} + x = 0$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + \frac{2}{t} y \end{cases} \Rightarrow \dot{\vec{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \vec{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 2/t \end{bmatrix}}_{B(t)} \vec{x}$$

h solutions: $\begin{cases} \sin t - t \cos t \\ \cos t + t \sin t \end{cases}$ both $\rightarrow \infty$ for $t \rightarrow \infty$

Theorem

$\dot{x} = Ax + B(t)x$. If:

- i) All eigenvalues of A are such that $\|\Phi(t)\| \leq C$
- ii) $\int_0^{\infty} \|B(t)\| dt < \infty$

Then $x=0$ is stable

Proof

$$\dot{x} = Ax, \quad \Phi(t) = e^{A(t-t_0)}; \quad x(t) = \Phi(t) \underline{z}$$

$\underbrace{\underline{z}}_{\text{constant vector}}$

$$\dot{x} = Ax + B(t)x$$

$$\text{Let } x(t) = \Phi(t) \underline{z}(t)$$

$\underbrace{\underline{z}(t)}_{\text{variation of constants}}$

$$\Phi \dot{z} + \dot{\Phi} z = A\Phi z + B(t)\Phi z$$

$\underbrace{\Phi \dot{z}}_{\text{variation of constants}}$

$$\Phi \dot{z} = B(t)\Phi z$$

$$\dot{z} = \Phi^{-1} B(t) \Phi z$$

Integrating from t_0 to t :

$$z(t) - z(t_0) = \int_{t_0}^t \Phi^{-1}(\tau) B(\tau) \Phi(\tau) z(\tau) d\tau$$

Multiply by $\Phi(t)$, and use $x = \Phi z$

$$x(t) - \Phi(t) \Phi^{-1}(t_0) x_0 = \Phi(t) \cdot \int_{t_0}^t \Phi^{-1}(\tau) B(\tau) x(\tau) d\tau$$

$$\left[\Phi(t) \Phi^{-1}(\tau) = e^{A(t-\tau)} \cdot e^{-A(\tau-t_0)} = e^{A(t-\tau+t_0-t_0)} = \Phi(t-\tau+t_0) \right]$$

$$\Phi^{-1}(t) = I$$

$$x(t) = \Phi(t) x_0 + \int_{t_0}^t \Phi(t-\tau+t_0) B(\tau) x(\tau) d\tau$$

$$\|x(t)\| \leq \underbrace{\|\Phi(t)\|}_{\text{Bounded}} \|x_0\| + \int_{t_0}^t \underbrace{\|\Phi(t-\tau+t_0)\|}_{\text{Bounded}} \underbrace{\|B(\tau)\|}_{\text{Bounded}} \|x(\tau)\| d\tau$$

$$\|x(t)\| \leq C \|x_0\| + \int_{t_0}^t C \|B(\tau)\| \|x(\tau)\| d\tau$$

Using Gronwall's Lemma:

$$\|x(t)\| \leq C \|x_0\| e^{C \int_{t_0}^t \|B(\tau)\| d\tau} < \infty$$

Theorem $\dot{x} = Ax + B(t)x$, if:

i) $\|\Phi(t)\| \leq Ce^{-\mu t}$ for some $\mu > 0$

ii) $\lim_{t \rightarrow \infty} \|B(t)\| = 0$

Then $x=0$ is asymptotically stable

Example

i) $\dot{x} = -x + \frac{1}{1+t^2} x$; $A = [-1]$; $B(t) = [\frac{1}{1+t^2}]$

↳ From 6.2: $x=0$ is stable

From 6.3: $x=0$ is asymptotically stable

ii) $\dot{x} = -x + \frac{1}{1+t} x$

↳ 6.2: not applicable

↳ 6.3: $x=0$ is asymptotically stable

iii) $\dot{x} = -x + \frac{a}{1+t} x$; $a > 0$, $t \geq 0$

↳ $\dot{x} = (a-1)x - \frac{a}{1+t} x$; $A = (a-1)$, $B(t) = (\frac{a}{1+t})$

↳ Before, $\|B(t)\| \neq 0$ for $t \rightarrow \infty$
Now it does

↳ If $-1+a < 0$, then $x=0$ is asymptotically stable
If $-1+a > 0$, $x=0$ is unstable

§6.3 $\dot{x} = A(t)x$ with $A(t) = A(t+T)$: Floquet Theory

Example i) $\dot{x} = x$, $x(t) = C \cdot e^t$

ii) $\dot{x} = \sin^2 t \cdot x$

$$\dot{x} = \left(\frac{1}{2} - \frac{1}{2} \cos(2t) \right) x$$

$$\hookrightarrow x(t) = C e^{\frac{1}{2}t} e^{\frac{1}{2} \sin(2t)}$$

Theorem $\dot{x} = A(t)x$ with $A(t) = A(t+T)$

$$\text{Then } \Phi(t) = P(t) e^{\frac{1}{T} B t}$$

$\hookrightarrow$ constant matrix
 $\hookrightarrow T$ -periodic matrix

Proof If $\Phi(t)$ is a fundamental matrix, then so is $\Phi(t+T)$

$$\tau = t + T$$

$$\frac{dx(\tau)}{d\tau} = \frac{dx(\tau)}{d\tau} \frac{d\tau}{dt} = \frac{dx(\tau)}{d\tau} = A(\tau)x(\tau) = A(t+T)x(t+T) = A(t)x(\tau)$$

There exists a nonsingular constant matrix C such that

$$\Phi(t+T) = \Phi(t) C$$

$$\text{Proof: } \Phi^{-1}(t) \Phi(t+T) = C$$

$$\det(1) \neq 0 \Rightarrow \det(C) \neq 0 \Rightarrow C \text{ non singular}$$

$$\begin{aligned} \frac{d}{dt} (\Phi^{-1}(t) \Phi(t+T)) &= \frac{d\Phi^{-1}(t)}{dt} \Phi(t+T) + \Phi^{-1}(t) \frac{d\Phi(t+T)}{dt} \\ &= -\Phi^{-1}(t) A(t) \Phi(t+T) + \Phi^{-1}(t) A(t+T) \Phi(t+T) \\ &= 0 \Rightarrow C \text{ constant} \end{aligned}$$

Let $C = e^{BT}$. Show that $\Phi(t) e^{-BT}$ is T -periodic.

$$P(t) = \Phi(t) e^{-BT}$$

$$P(t+T) = \Phi(t+T) e^{-B(t+T)}$$

$$= \Phi(t) C e^{-Bt} e^{-BT}$$

$$= \Phi(t) e^{-BT} = P(t)$$

$$\hookrightarrow \Phi(t) = P(t) e^{BT}$$

$$\Phi(t+T) = P(t) C$$

$\hookrightarrow$ monodromy matrix. λ_i are characteristic multipliers

Let $\vec{s}$ be an eigenvector and ρ its corresponding eigenvalue

$$\vec{x}(t) = \vec{s} \Rightarrow \vec{x}(t+T) = \Phi(t+T) \vec{s} = \Phi(t) C \vec{s} = \Phi(t) \rho \vec{s} = \rho \vec{x}(t)$$

LECTURE 9 Floquet Stability & Stability

§6.3

$$\Phi(t) = P(t)e^{Bt}$$

$$\dot{x} = A(t)x; \quad A(t+T) = A(t)$$

Theorem

$$\rho_i = e^{\lambda_i T}$$

λ_i Eigenvalue of B
 ρ_i Eigenvalue of Φ

$$\rho_1 \rho_2 \dots \rho_n = e^{\int_0^T \text{Tr}(A(s)) ds}$$

$$\sum \lambda_i = \frac{1}{T} \int_0^T \text{Tr}(A(s)) ds \quad \text{mod} \left(\frac{2\pi i}{T} \right) \quad > 0 \rightarrow \text{unstable}$$

Proof

$$\Phi(0) = I$$

$$\text{Det}(\Phi(t+\delta)) = \text{Det}(\Phi(t) + \delta \Phi'(t) + O(\delta^2))$$

$$= \text{Det}((I + \delta A(t) + O(\delta^2)) \Phi(t))$$

$$= \text{Det}(I + \delta A(t) + O(\delta^2)) \cdot \text{Det}(\Phi(t))$$

$$= (1 + \delta \text{Tr}(A(t)) + O(\delta^2)) \cdot \text{Det}(\Phi(t))$$

$$\text{Det}(\Phi(t+\delta)) - \text{Det}(\Phi(t)) = (\delta \text{Tr}(A(t)) + O(\delta^2)) \text{Det}(\Phi(t))$$

$\hookrightarrow$ Divide by δ and take $\lim_{\delta \rightarrow 0}$:

$$\frac{d}{dt} \text{Det}(\Phi(t)) = \text{Tr}(A(t)) \text{Det}(\Phi(t))$$

$$\text{Det}(\Phi(t)) = \text{Det}(\underbrace{\Phi(0)}_{I}) \cdot e^{\int_0^t \text{Tr}(A(s)) ds}$$

$$\text{Det}(\Phi(T)) = e^{\int_0^T \text{Tr}(A(s)) ds}$$

$$\text{Det}(C) = e^{\int_0^T \text{Tr}(A(s)) ds}$$

$$\prod \rho_i = e^{\int_0^T \text{Tr}(A(s)) ds}$$

Product of eigenvalues is the determinant

$$\begin{aligned} \text{First} &= 1 + \delta(a_{11} + a_{22} + \dots) + O(\delta^2) \\ &= 1 + \delta \text{Tr}(A(t)) + O(\delta^2) \end{aligned}$$

Chapter 7: Stability by Linearization

Theorem Poincaré - Lyapunov

$$\dot{x} = A x + B(t)x + f(t, x); \quad x \in \mathbb{R}^n, \quad A^{(n \times n)}, \quad B^{(n \times n)}, \quad f \in \mathbb{R}^n$$

i) A is a constant $n \times n$ matrix for which all eigenvalues have negative real parts

ii) $B(t)$ is a continuous $n \times n$ matrix with $\lim_{t \rightarrow \infty} \|B(t)\| = 0$

iii) f is a nonlinear function continuous in t and Lipschitz continuous

in x with $\lim_{x \rightarrow 0} \|f(t, x)\| / \|x\| = 0 \rightarrow$ No linear terms in f .

Then, $\|x(t)\| \leq C \cdot \|x_0\| e^{-\mu(t-t_0)}$ for $t > t_0$, $\mu > 0$ exponential decay

and $x=0$ is asymptotically stable

Theorem Instability of the trivial solution

$$\dot{x} = Ax + B(t)x + f(t, x)$$

i) A has at least 1 eigenvalue with $\operatorname{Re}(\lambda) > 0$

$$\text{ii) } \lim_{t \rightarrow \infty} \|B(t)\| = 0$$

$$\text{iii) } \lim_{\|x\| \rightarrow 0} \frac{\|f(t, x)\|}{\|x\|} = 0$$

Then, $x=0$ is unstable

$$x = Sy$$

time-varying

$$\dot{x} = \dot{S}y = ASy + B(t)Sy + f(t, Sy)$$

$$\hookrightarrow \dot{y} = \underbrace{S^{-1}AS}_{\text{Assumed diagonal}} y + S^{-1}B(t)Sy + S^{-1}f(t, Sy)$$

$$\operatorname{Re}(\lambda_i) \geq \sigma > 0, \quad i = 1, \dots, k$$

$$\operatorname{Re}(\lambda_i) < 0, \quad i = k+1, \dots, n$$

$$R^2 = \sum_{i=1}^k |y_i|^2, \quad r^2 = \sum_{i=k+1}^n |y_i|^2$$

$$\frac{d}{dt} |y_i|^2 = \frac{d}{dt} (y_i \bar{y}_i) = \dot{y}_i \bar{y}_i + y_i \dot{\bar{y}}_i$$

$$= (S^{-1}ASy)_i \bar{y}_i + (S^{-1}B(t)Sy)_i \bar{y}_i + (S^{-1}f(t, Sy))_i \bar{y}_i$$

$$+ y_i \overline{(S^{-1}ASy)_i} + y_i \overline{(S^{-1}B(t)Sy)_i} + y_i \overline{(S^{-1}f(t, Sy))_i}$$

$$= 2 \operatorname{Re}(\lambda_i) y_i \bar{y}_i + (S^{-1}B(t)Sy)_i \bar{y}_i + \dots + y_i \overline{(S^{-1}f(t, Sy))_i}$$

$$\left| \sum_{i=1}^k (S^{-1}B(t)Sy)_i \bar{y}_i \right| \leq \sum_{i=1}^k |y_i| \bar{y}_i \quad \text{for large } t$$

$$\sum_{i=1}^k |S^{-1}f(t, Sy)_i| \bar{y}_i \leq \sum_{i=1}^k |y_i| \bar{y}_i \quad \text{for } y_i \text{ close enough to the eq. point}$$

Similarly for $\sum_{i=k+1}^n$

$$\frac{dR^2}{dt} \geq 2 \sum_{i=1}^k (\operatorname{Re}(\lambda_i) - 2\varepsilon) |y_i|^2$$

$$\frac{dr^2}{dt} \leq 2 \sum_{i=k+1}^n (\operatorname{Re}(\lambda_i) + 2\varepsilon) |y_i|^2$$

Now we choose $0 < \varepsilon < \frac{1}{n} \sigma$

Then: $\operatorname{Re}(\lambda_i) - 2\varepsilon \geq 2\varepsilon$ for $i = 1, \dots, k$
 $\operatorname{Re}(\lambda_i) + 2\varepsilon \leq 2\varepsilon$ for $i = k+1, \dots, n$

Then: $\frac{1}{2} \frac{d(R^2 - r^2)}{dt} \geq 2\varepsilon (R^2 - r^2)$

Now take as initial condition $R^2 - r^2 = a > 0$ for $t = t_0$

$R^2 - r^2 \geq a e^{4\varepsilon(t-t_0)}$ for $t > t_0$

$\|y\|^2 \geq R^2 - r^2 > a e^{4\varepsilon(t-t_0)} \rightarrow \text{unstable}$

Chapter 8: Stability Analysis by the Direct Method (Lyapunov Functions)

$$\begin{cases} \dot{x} = ax - y + kx(x^2 + y^2) \\ \dot{y} = x - ay + ky(x^2 + y^2) \end{cases}$$

with a, k constants, $a > 0$

Finding eq. points:

$$y = ax + kx(x^2 + y^2)$$

$$x - a(ax + kx(x^2 + y^2)) + k(ax + kx(x^2 + y^2))(x^2 + y^2) = 0$$

$$\hookrightarrow \begin{cases} y = ax + kx(x^2 + y^2) \\ x = 0 \vee x = 1 - a + k^2(x^2 + y^2) = 0 \end{cases}$$

$$\hookrightarrow y=0 \quad k(x^2 + y^2)^2 = \frac{a^2 - 1}{k^2} \rightarrow \text{No solutions for } a^2 - 1 < 0$$

$$\begin{cases} a^2 = 1 : x = y = 0 \\ a^2 > 1 : \text{circle, } r = \sqrt{\frac{a^2 - 1}{k^2}} \end{cases}$$

$\hookrightarrow$ From first condition:

$$y = ax + kx \sqrt{\frac{a^2 - 1}{k^2}}$$

$\hookrightarrow$ line $\rightarrow 2$ points (+ origin)

LECTURE 10 Chapter 8: Lyapunov Functions

Example

$$\begin{cases} \dot{x} = ax - y + kx(x^2 + y^2) \\ \dot{y} = ay - x + ky(x^2 + y^2) \end{cases}$$

a, k constants
 $a > 0$

What can be said about the stability of $(0,0)$?

Linearizing:

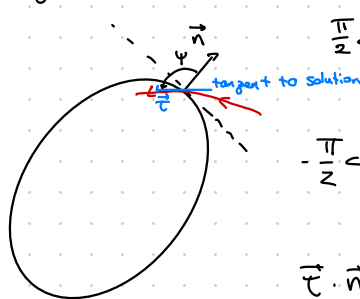
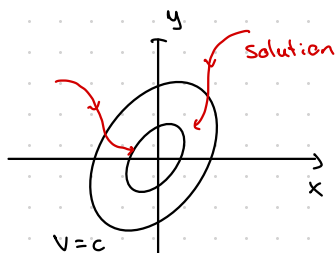
$$\begin{cases} \dot{x} = ax - y \\ \dot{y} = ay - x \end{cases} \rightarrow \text{eigenvalues: } \lambda^2 = a^2 - 1$$

$\hookrightarrow a > 1$: $\lambda_{1,2} = \pm \sqrt{a^2 - 1} \rightarrow$ Saddle point

$a = 1$: $\lambda_{1,2} = 0 \rightarrow$ degenerate (line of eq.)

$0 < a < 1$: $\lambda_{1,2} = \pm i\sqrt{1-a^2}$ Real part 0, no conclusion for nonlin. system

Introduce: $V(x,y) = x^2 - 2axy + y^2 = \text{constant}$ (for $0 < a < 1$)
 $\hookrightarrow$ First integral for $k=0$



$\frac{\pi}{2} < \psi < \frac{3\pi}{2}$: for all solutions crossing $V=c$
 $\hookrightarrow$ stability

$-\frac{\pi}{2} < \psi < \frac{\pi}{2}$: for all solutions crossing $V=c$
 $\hookrightarrow$ instability

$$\vec{\tau} \cdot \vec{n} = \|\vec{\tau}\| \|\vec{n}\| \cos(\angle(\vec{\tau}, \frac{\nabla V}{\|\nabla V\|}))$$

$$\vec{\tau} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \quad \vec{n} = \begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \end{pmatrix} \cdot \frac{1}{\|\nabla V\|}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \end{pmatrix} \cdot \frac{1}{\|\nabla V\|} = \|\vec{\tau}\| \|\nabla V\| \cos(\angle(\vec{\tau}, \frac{\nabla V}{\|\nabla V\|}))$$

$$\dot{x} \frac{\partial V}{\partial x} + \dot{y} \frac{\partial V}{\partial y} = \|\vec{\tau}\| \|\nabla V\| \cos(\angle(\vec{\tau}, \frac{\nabla V}{\|\nabla V\|}))$$

$$\text{Stability } (\frac{\pi}{2} < \psi < \frac{3\pi}{2}): \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial y} \dot{y} < 0$$

$\Leftrightarrow$

$$\hookrightarrow V(x,y) < 0$$

$$\text{Instability } (-\frac{\pi}{2} < \psi < \frac{\pi}{2}): \frac{\partial V}{\partial x} \dot{x} + \frac{\partial V}{\partial y} \dot{y} > 0$$

$$\hookrightarrow V(x,y) > 0$$

$$\hookrightarrow V = 2k(x^2 + y^2)(x^2 + y^2 - 2axy) \rightarrow \text{stability determined by } k$$

$\hookrightarrow k \leq 0$: stable
 $k > 0$: unstable

§8.2 $\dot{x} = f(t, x)$ with $f(t, 0) = 0$

Definition $V(t, x)$ is continuously differentiable

$$V(t, 0) = 0$$

Definition $V(x)$ is called positively or negatively definite if $V(x) > 0$ or $V(x) < 0$ for $x \in D \setminus \{0\}$ respectively

$V(t, x)$ is called positively or negatively definite if there exists a function $W(x)$ with $W(0) = 0$ such that $0 < W(x) \leq V(t, x)$ or $V(t, x) \leq W(x) < 0$ for $x \in D \setminus \{0\}$ respectively

Definition "Positively semi-definite": replace > 0 by ≥ 0

Example $D = \{x \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$

$$V_1(x, y, z) = x^2 + 2y^2$$

$$V_2(x, y, z) = x^2 + 2y^2 + 3z^2 + z^3$$

$$V_3(t, x, y, z) = x^2 + y^2 + \cos(t) \cdot z^2$$

$$V_4(x, y, z) = x^2 + 2x^3 + y^2 + z^2$$

P.D.	P.S.D.	N.D.	N.S.D.
No	Yes	No	No
Yes	Yes	No	No
No	No	No	No
No	No	No	No

Orbital derivative: $L_t V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} \frac{dx_1}{dt} + \dots + \frac{\partial V}{\partial x_n} \frac{dx_n}{dt}$

$$V(t, x) = \frac{\partial V}{\partial t} + \underbrace{\frac{\partial V}{\partial x}}_{\text{Vector}} \cdot f(t, x)$$

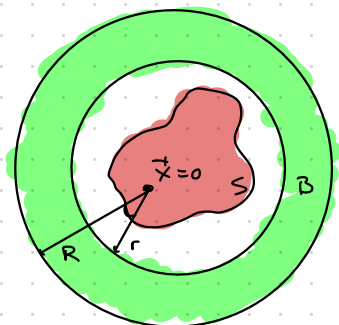
$$\dot{x} = f(t, x)$$

Theorem $\dot{x} = f(t, x)$ with $f(t, 0) = 0$

8.1

If a function $V(t, x)$ can be found in a neighborhood of $x=0$ and positively definite for $t \geq t_0$, $V(t, 0) = 0$ and $L_t V \leq 0$, then $x=0$ is Lyapunov stable

Proof



$$\text{in } S: V(t, x) \geq m$$

$$\text{Shell } B: 0 < r \leq \|x\| \leq R$$

$$V(t, x(t)) - V(t_0, x(t_0)) = \int_{t_0}^t L_\tau V(\tau, x) d\tau \leq 0$$

Theorem $\dot{x} = f(t, x)$ with $f(t, 0) = 0$

8.2

$V(t, x)$ is positively definite outside $x=0$ (in $D \setminus \{0\}$)

$L_t V < 0$ in D . Then, $x=0$ is asymptotically stable

Proof

Let's assume that $\|x\| > a$

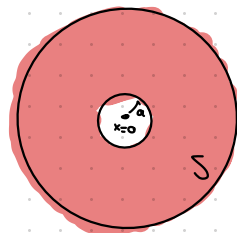
$$V(t, x(t)) - V(t_0, x(t_0)) = \int_{t_0}^t L_t V(\tau, x) d\tau \leq -\mu(t - t_0)$$

For $\|x\| > a$ we have:

$$L_t V \leq -\mu \text{ for a certain } \mu > 0$$

For t large: $V(t, x)$ becomes negative

↳ Thus $\|x\| > a$ is not true (as V is pos. def.)



Theorem $\dot{x} = f(t, x)$ with $f(t, 0) = 0$

8.3

$V(t, x)$ is defined in the neighborhood of $x=0$, satisfying:

a) $V(t, x) \rightarrow 0$ for $\|x\| \rightarrow 0$ uniformly in t

b) $L_t V$ is positively definite in a neighborhood of $x=0$

c) From a certain $t = t_1 \geq t_0$, $V(t, x)$ takes positive values in each sufficiently small neighborhood of $x=0$ ↳ not everywhere !

Then, $x=0$ is unstable

Proof We have positive constants a and b .

for $x \neq 0$ and $\|x\| \leq a$: $L_t V(t, x) \geq W(x) > 0$

and $|V(t, x)| \leq b$

Assume that $x=0$ is stable $\Rightarrow$ starting with $x(t_0) = 0 \in \mathcal{E}$, then $\|x(t_0)\| \leq a$ for $t \geq t_1$

Let x_0 be such that $V(t_1, x(t_1)) > 0 \Leftrightarrow V(t_1, x_0) > 0$

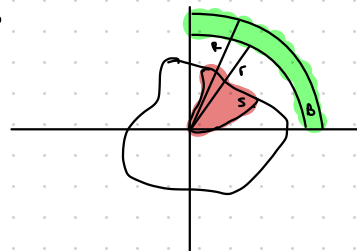
$$V(t, x(t)) - V(t_1, x_0) = \int_{t_1}^t L_t V(\tau, x) d\tau > 0$$

$V(t, x)$ is increasing.

In S positive values of V

$$0 < r \leq \|x\| \leq R$$

$$L_t V > \mu, \mu > 0$$



$$V(t, x) - V(t_1, x_0) > \mu(t - t_0)$$

V grows unboundedly

V is continuous

V is defined on a bounded domain $\longrightarrow V$ is bounded on the domain

$\hookrightarrow x=0$ is not stable

§8.3 Hamiltonian Systems, First Integrals

$$\begin{cases} \dot{p}_i = -\frac{\partial H}{\partial q_i} \\ \dot{q}_i = \frac{\partial H}{\partial p_i} \end{cases} \quad i = 1, 2, \dots, n$$

$$L_t H = \underbrace{\frac{\partial H}{\partial p_i} \frac{dp_i}{dt}}_{\underbrace{\frac{\partial H}{\partial q_i}}_{\underbrace{\frac{\partial H}{\partial p_i}}}} + \underbrace{\frac{\partial H}{\partial q_i} \frac{dq_i}{dt}}_{\underbrace{\frac{\partial H}{\partial p_i}}} = 0$$

$$V(p, q) = H(p, q) - H(0, 0) \rightarrow V(0, 0) = 0$$

$(p, q) = (0, 0)$ is an equilibrium point

$$L_t V = L_t H = 0$$

Is $V(p, q) > 0$ for $(p, q) \neq (0, 0)$?

Theorem 8.4 If $H(p, q) - H(0, 0)$ is sign definite, then $(p, q) = (0, 0)$ is Lyapunov stable

LECTURE 11 Chapter 8 + 9

§8.3

$$H(p, q) = \frac{1}{2} \sum_{i=1}^n p_i^2 + \phi(q)$$

$$\begin{cases} \dot{p}_i = -\frac{\partial H}{\partial q_i} = -\frac{\partial \phi(q)}{\partial q_i} \\ \dot{q}_i = \frac{\partial H}{\partial p_i} = p_i \end{cases} \Rightarrow \ddot{q}_i = \dot{p}_i = -\frac{\partial \phi(q)}{\partial q_i}$$

Equilibrium points: $\begin{cases} p_i = 0 \\ \frac{\partial \phi(q)}{\partial q_i} = 0 \end{cases} \Rightarrow$ we can get them from $\phi(q)$

Assume in our equilibrium point $\phi(q)$ has a minimum in (p_{eq}, q_{eq}) what can we then say about the stability of this eq. point

$$V(p, q) = H(p, q) - H(p_{eq}, q_{eq}) \rightarrow V(p_{eq}, q_{eq}) = 0; \text{ outside } V(p, q) > 0$$

$$L_t V = L_t (H(p, q) - H(p_{eq}, q_{eq})) = L_t H(p, q) = 0$$

$\hookrightarrow (p_{eq}, q_{eq})$ is Lyapunov stable

Let's now assume that $\phi(q)$ has a minimum in the eq. point $(0, 0)$ can always be done with translation

$$\phi(q) = -\sum_{i=1}^n a_i q_i^{2m} + O(\|q\|^{2m+1}) \text{ for } \|q\| \rightarrow 0 \text{ with } a_i > 0, m \geq 1$$

$$V(p, q) = \sum_{i=1}^n p_i \cdot q_i$$

From theorem 8.3, i) and iii) are satisfied. ii):

$$\begin{aligned} L_t V &= \sum_{i=1}^n \dot{p}_i \cdot q_i + p_i \cdot \dot{q}_i = \sum_{i=1}^n -\frac{\partial \phi(q)}{\partial q_i} q_i + p_i^2 \\ &= \sum_{i=1}^n 2ma_i q_i^{2m} + O(\|q\|^{2m}) q_i + p_i^2 > 0 \text{ sufficiently close to } (0, 0) \end{aligned}$$

$\hookrightarrow$ thus, $(0, 0)$ is unstable

If I_k is a first integral of $\dot{x} = f(t, x)$ for $k = 1, \dots, m$; $L_t I_k = 0$

Let $f = f(I_1, I_2, \dots, I_m)$. Then $f = \text{constant}$ is also a first integral

$$L_t f = \sum_{i=1}^m \frac{\partial f}{\partial I_i} \cdot \frac{dI_i}{dt} = 0$$

Example

$$\begin{cases} A\dot{x} = (B-C)yz \\ B\dot{y} = (C-A)xz \\ C\dot{z} = (A-B)xy \end{cases}$$

with $0 < A < B < C$. what can be said about the stability of the eq. point $(x_0, 0, 0)$, with $x_0 > 0$

$$\frac{A}{B} \frac{dx}{dy} = \frac{B-C}{C-A} \frac{y}{x} \Rightarrow A(C-A)x dx = B(B-C)y dy$$

$$\frac{A}{2}(C-A)x^2 = \frac{B}{2}(B-C)y^2 + \text{const.}$$

$$L_1 V_1(x, y) = A(C-A)x^2 + B(C-B)y^2$$

$$\text{Similarly: } V_2(y, z) = B(B-A)y^2 + C(C-A)z^2$$

$$\text{We know } L_1 V_1 = L_1 V_2 = 0$$

$$V(x, y, z) = \underbrace{(V_1(x, y) - V_1(x_0, 0))}_{\substack{\text{is 0 in } (x_0, 0, 0) \\ \text{sgn definite}}} + V_2(y, z) \rightarrow \text{Lyapunov function}$$

$$\text{Observe } V(x_0, 0, 0) = 0 \text{ and } V > 0 \text{ for } \vec{x} \in D \setminus \left\{ \begin{pmatrix} x_0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$L_1 V = 0 \xrightarrow{B^1} (x_0, 0, 0) \text{ is stable}$$

Chapter 9 Introduction to Perturbation Methods

$$f(t, x; \varepsilon) = f(t, x; 0) + \varepsilon f_1(t, x) + \varepsilon^2 f_2(t, x) + \dots + \varepsilon^n f_n(t, x)$$

$$0 < \varepsilon \ll 1$$

$1, \varepsilon, \varepsilon^2, \dots, \varepsilon^n$ are called order functions &

$$f(t, x; \varepsilon) = \sum_{n=0}^N \delta_n(\varepsilon) f_n(t, x)$$

$\delta_n(\varepsilon)$ are order functions. Properties:

i) $\delta(\varepsilon)$ is continuous and positive in $(0, \varepsilon_0]$

ii) $\lim_{\varepsilon \rightarrow 0} \delta(\varepsilon)$ exists

iii) $\delta(\varepsilon)$ is monotonically decreasing in ε as ε tends to 0

Definition a) $\delta_1(\varepsilon) = O(\delta_2(\varepsilon))$ as $\varepsilon \rightarrow 0$: there exists a constant μ such that $\delta_1(\varepsilon) \leq \mu \delta_2(\varepsilon)$

b) $\delta_1(\varepsilon) = o(\delta_2(\varepsilon))$ as $\varepsilon \rightarrow 0$: $\lim_{\varepsilon \rightarrow 0} \frac{\delta_1(\varepsilon)}{\delta_2(\varepsilon)} = 0$

Example i) $\varepsilon^4 = O(\varepsilon^2)$? $\rightarrow$ True

ii) $\varepsilon = o(1)$? $\rightarrow$ True

iii) $\varepsilon t \sin x = O(\varepsilon)$? $\rightarrow$ True if $t = O(1)$
False if $t = O(1/\varepsilon)$

iv) $\varepsilon^n = O(\varepsilon)$ for $n \geq 2$? $\rightarrow$ True

v) $e^{-\frac{1}{\varepsilon}} = O(\varepsilon)$? $\rightarrow$ True

vi) for all $0 < \varepsilon < 1$, as $\varepsilon \rightarrow 0$: $\varepsilon^{-\frac{1}{2}} \leq \varepsilon^n$

§9.3. Naive Expansions

$$\dot{x} = f(t, x; \varepsilon)$$

$$f(t, x; \varepsilon) = f(t, x; 0) + \varepsilon f_1(t, x) + \dots$$

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \dots$$

Theorem 9.1 $\|x(t) - (x_0(t) + \varepsilon x_1(t) + \dots + \varepsilon^n x_n(t))\| = O(\varepsilon^{n+1})$ for $t = O(1)$

Chapter 10 Poincaré - Lindstedt Method

Accurate approximations of periodic solutions

$$\ddot{x} + x = \varepsilon f(x, \dot{x}; \varepsilon)$$

§10.1: autonomous systems

$$x(t) = x_0(t) + \varepsilon x_1(t) + \dots$$

$$O(1): \ddot{x}_0 + x_0 = 0; \quad \varepsilon = 0 \text{ problem: } x_0(t) = a_0 \cos t + b_0 \sin t$$

Let $T(\varepsilon)$ be a period of the equation. Then $T(0) = 2\pi$

$$\ddot{x} + x = \varepsilon f(x, \dot{x}; \varepsilon)$$

$$x(0) = a(\varepsilon)$$

$$\dot{x}(0) = 0$$

$$\lim_{\varepsilon \rightarrow 0} x(t) = a(0) \cos t$$

$$T(\varepsilon) = T_0 + \varepsilon T_1 + \dots$$

$\rightarrow 2\pi$ periodic in θ

time transformation: $\omega(\varepsilon)t = \theta$ such that $x(t) = \tilde{x}(\theta)$

LECTURE 72 Poincaré-Lindstedt Method, Continualization Method for Periodic Solutions Ch. 10

Ch. 11 Introduction to the averaging method

Ex 10.1 $\ddot{x} + x = \varepsilon f(x, \dot{x}; \varepsilon)$

Assume a periodic solution exists: $x(t) = x(t+T)$
 T is the period

From Ch. 9: Poincaré-expansion theorems

On a line-scale of $O(1)$: $x(t) = x_0(t) + \varepsilon x_1(t) + \dots$

$O(1)$ problem: $\ddot{x}_0 + x_0 = 0 \rightarrow x_0(t) = a_0 \cos t + b_0 \sin t$

$$T = T(\varepsilon) = 2\pi + O(\varepsilon)$$

$$\text{Now consider: } \begin{cases} \ddot{x} + x = \varepsilon f(x, \dot{x}; \varepsilon) \\ x(0) = a(\varepsilon) \\ \dot{x}(0) = c \end{cases}$$

$$a(\varepsilon) = a_0 + \varepsilon a_1 + \dots$$

$$T(\varepsilon) = T_0 + \varepsilon T_1 + \dots$$

$$\lim_{\varepsilon \rightarrow 0} x(t) = a_0 \cos t$$

Transformation: $x(\varepsilon) = \tilde{x}(\theta)$ with $\omega(\varepsilon)t = \theta$
 $\omega(\varepsilon)$ unknown, related to the period
 $\tilde{x}$ 2π -periodic in θ

$$\frac{1}{\omega(\varepsilon)} = 1 - \varepsilon \eta(\varepsilon), \quad \omega(\varepsilon) = \omega_0 + \varepsilon \omega_1 + \dots$$

$$\frac{d\tilde{x}(t)}{d\varepsilon} = \frac{d\tilde{x}(\theta)}{d\theta} \cdot \frac{d\theta}{d\varepsilon} = \tilde{x}'(\theta) \omega(\varepsilon)$$

$$\frac{d^2 x(t)}{dt^2} = \frac{d}{d\varepsilon} \left(\frac{d\tilde{x}(\theta)}{d\theta} \right) = \frac{d}{d\varepsilon} (\tilde{x}'(\theta) \omega(\varepsilon)) = \frac{d}{d\theta} (\tilde{x}'(\theta) \omega(\varepsilon)) \frac{d\theta}{d\varepsilon} = \tilde{x}''(\theta) \omega^2(\varepsilon)$$

ODE becomes:

$$\omega^2(\varepsilon) \tilde{x}''(\theta) + \tilde{x}(\theta) = \varepsilon f(\tilde{x}(\theta), \omega(\varepsilon) \tilde{x}'(\theta); \varepsilon)$$

$$\text{or } \tilde{x}''(\theta) + \frac{1}{\omega^2(\varepsilon)} \tilde{x}(\theta) = \frac{\varepsilon}{\omega^2(\varepsilon)} f(\tilde{x}(\theta), \omega(\varepsilon) \tilde{x}'(\theta); \varepsilon)$$

$$\text{or } \tilde{x}''(\theta) + (1 - \varepsilon \eta(\varepsilon)) \tilde{x}(\theta) = \varepsilon (1 - \varepsilon \eta(\varepsilon)) f(\tilde{x}(\theta), \omega(\varepsilon) \tilde{x}'(\theta); \varepsilon)$$

$$\text{or } \tilde{x}'' + \tilde{x} = \varepsilon (\eta(\varepsilon) \tilde{x} + (1 - \varepsilon \eta(\varepsilon)) f(\dots)) = \varepsilon g(\tilde{x}, \tilde{x}'; \varepsilon, \eta)$$

$$\Gamma x''(t) + x(t) = f(t); \quad x(0) = a, \quad x'(0) = 0$$

Method of variation of constants:

$$\hookrightarrow x(t) = a \cos t + \int_0^t \sin(t-\tau) f(\tau) d\tau$$

$$x(\theta) = a \cos \theta + \varepsilon \int_0^\theta \sin(\theta-\tau) g(x(\tau), x'(\tau); \varepsilon, \eta) d\tau \quad \text{integral equation}$$

What about the 2π -periodicity?

$$x(\theta) = a \cos \theta + \varepsilon \int_{\theta+2\pi}^\theta \sin(\theta-\tau) g(x(\tau), x'(\tau); \varepsilon, \eta) d\tau$$

$$x(\theta+2\pi) = a \cos(\theta+2\pi) + \varepsilon \int_{\theta+2\pi}^{\theta+2\pi} \sin(\theta+2\pi-\tau) g(\dots) d\tau$$

$$x(\theta+2\pi) - x(\theta) = \varepsilon \int_{\theta+2\pi}^\theta \sin(\theta-\tau) g(\dots) d\tau = 0$$

$$\hookrightarrow \int_0^{2\pi} \sin(\theta-\tau) g(\dots) d\tau = 0$$

$\hookrightarrow \sin \cos - \cos \sin$

$$\hookrightarrow \sin \varepsilon \int_0^{2\pi} \cos \tau g(\dots) d\tau - \cos \varepsilon \int_0^{2\pi} \sin \tau g(\dots) d\tau = 0$$

$$\hookrightarrow \begin{cases} \int_0^{2\pi} \sin \tau g(\dots) d\tau = 0 \\ \int_0^{2\pi} \cos \tau g(\dots) d\tau = 0 \end{cases} \Rightarrow \begin{cases} F_1(a(\varepsilon), \eta(\varepsilon)) = 0 \\ F_2(a(\varepsilon), \eta(\varepsilon)) = 0 \end{cases} \quad \begin{array}{l} \text{implicit} \\ \text{function} \\ \text{theorem} \end{array}$$

$$\Gamma g = \eta x + (1-\varepsilon\eta) f(x, (1-\varepsilon\eta)^{-1/2} x; \varepsilon)$$

$$\underline{\varepsilon=0} : \begin{cases} \int_0^{2\pi} \sin \tau (\eta_0 a_0 \cos \tau + f(a_0 \cos \tau, -a_0 \sin \tau; \varepsilon)) d\tau = 0 \\ \int_0^{2\pi} \cos \tau (\eta_0 a_0 \cos \tau + f(a_0 \cos \tau, -a_0 \sin \tau; \varepsilon)) d\tau = 0 \end{cases}$$

$$\hookrightarrow \begin{cases} \int_0^{2\pi} \sin \tau f(a_0 \cos \tau, -a_0 \sin \tau; \varepsilon) d\tau = 0 \\ \pi \eta_0 a_0 + \int_0^{2\pi} \cos \tau f(a_0 \cos \tau, -a_0 \sin \tau; \varepsilon) d\tau = 0 \end{cases}$$

Assume $f(x, \dot{x}) = (1-x^2)\dot{x}$

$$\hookrightarrow \begin{cases} \int_0^{2\pi} \sin \tau (1-a_0^2 \cos^2 \tau) (-a_0 \sin \tau) d\tau = 0 \\ \pi \eta_0 a_0 + \int_0^{2\pi} \cos \tau (1-a_0^2 \cos^2 \tau) (-a_0 \sin \tau) d\tau = 0 \end{cases}$$

$$\hookrightarrow \begin{cases} \frac{1}{2} a_0 (1 - \frac{1}{4} a_0^2) = 0 \\ \pi \eta_0 a_0 = 0 \end{cases} \Rightarrow \begin{cases} a_0 = 2 \\ \eta_0 = 0 \end{cases} \quad \begin{array}{l} \text{non-trivial} \\ \text{solution} \end{array}$$

For autonomous equations (algorithm):

$$\theta = \omega(\varepsilon)t$$

$$\omega(\varepsilon) = (1 - \varepsilon\eta)^{-1/2}$$

$$x(t) = \tilde{x}(\theta)$$

$$\begin{cases} \tilde{x}(\theta) = a_0 \cos \theta + \sum_{n=1}^{\infty} \varepsilon^n \tilde{x}_n(\theta) \\ a = a_0 + \varepsilon a_1 + \dots \\ \eta = \eta_0 + \varepsilon \eta_1 + \dots \end{cases}$$

$\tilde{x}_n$ 2π -periodic in θ

Van der Pol equation:

$$x'' + x = \varepsilon (\eta x + (1 - \varepsilon\eta)^{1/2} (1 - x^2) x')$$

$$O(\varepsilon): \tilde{x}_1'' + \tilde{x}_1 = \underbrace{a_0 \eta_0 \cos \theta}_{\substack{\downarrow \eta=0 \\ \text{else not } 2\pi\text{-periodic}}} + \underbrace{a_0 \left(\frac{1}{4} a_0^2 - 1 \right) \sin \theta}_{\substack{\downarrow \eta=0}} + \frac{1}{4} a_0^3 \sin(3\theta) - \frac{a_0^3}{3^2} \sin(3\theta)$$

$$\tilde{x}_1(\theta) = A_1 \cos \theta + B_1 \sin \theta - \frac{a_0^3}{3^2} \sin(3\theta)$$

$$\tilde{x}_1(0) = a_1, \tilde{x}_1'(0) = 0$$

$$O(\varepsilon^2): \tilde{x}_2'' + \tilde{x}_2 = \dots \quad \tilde{x}_2(\theta) = a_2 \cos \theta + \frac{3}{4} a_1 \sin \theta - \frac{1}{4} a_1 \sin(3\theta)$$

§10.4 Nonautonomous equation (algorithm)

Example $\ddot{x} + \varepsilon \mu \dot{x} + x - \varepsilon x^3 = \varepsilon h \cos(\omega t)$

known: μ, h , and ω ($\frac{1}{\omega^2} = 1 - \varepsilon \beta$, β known)

What can we say about periodic solutions?

Look for 2π -periodic solutions in θ ; $x(t) = \tilde{x}(\theta)$

ICs: $x(0) = a(\varepsilon)$, $\dot{x}(0) = 0$; we know the possible periods: $\frac{2\pi}{\omega} \cdot k$

Let's look for $\frac{2\pi}{\omega}$ -periodic solutions

$$\theta = \omega t + \psi(\varepsilon)$$

$\downarrow$ known $\downarrow$ unknown phase; for autonomous not important cause we can translate in time

$\downarrow$ unknowns: $a(\varepsilon)$ and $\psi(\varepsilon)$

$$\begin{cases} \tilde{x}(\theta) = a_0 \cos \theta + \sum_{n=1}^{\infty} \varepsilon^n \tilde{x}_n(\theta) \\ a(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n a_n \\ \psi(\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \psi_n \end{cases}$$

$$x'' + x = \varepsilon g(x, x', \psi, \theta, \varepsilon)$$

$$= \varepsilon \beta x - \varepsilon \mu (1 - \varepsilon \beta)^{1/2} x' + \varepsilon (1 - \varepsilon \beta) x^3 + \varepsilon (1 - \varepsilon \beta) h \cos(\theta - \psi)$$

$$O(\varepsilon): \tilde{x}_1'' + \tilde{x}_1 = \beta a_0 \cos \theta + \mu a_0 \sin \theta + a_0^3 \cos^3 \theta + h \cos(\theta - \psi_0)$$

$\downarrow \cos \theta \cos \psi_0 + \sin \theta \sin \psi_0$

$$= \underbrace{(\mu a_0 + h \sin \psi_0)}_{\psi=0} \sin \theta + \underbrace{(\beta a_0 + h \cos \psi_0 + \frac{3}{u} a_0^3)}_{\psi=0} \cos \theta + \frac{a_0^3}{u} \cos(3\theta)$$

else not periodic

$$\hookrightarrow f_1(\theta) = A_1 \cos \theta + B_1 \cos \theta - \frac{a_0^3}{32} \cos(3\theta)$$

How to solve

$$\begin{cases} \mu a_0 + h \sin \psi_0 = 0 \\ \beta a_0 + h \cos \psi_0 + \frac{3}{u} a_0^3 = 0 \end{cases}$$

$$\hookrightarrow \text{square and add: } h^2 = (\mu a_0)^2 + \left(\beta a_0 + \frac{3}{u} a_0^3 \right)^2$$

$\hookrightarrow$ cubic in $a_0^2 \rightarrow$ Cardano

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Chapter 11 Averaging

$$\ddot{x} + x = \varepsilon f(x, \dot{x})$$

$$\underline{\varepsilon = 0}: x(t) = r_0 \cos(t + \psi_0)$$

$$\dot{x}(t) = -r_0 \sin(t + \psi_0)$$

$$\underline{\varepsilon \neq 0}: x(t) = r(t) \cos(t + \psi(t)) \quad \text{variation of constants}$$

$$\dot{x}(t) = -r(t) \sin(t + \psi(t))$$

Starting with one unknown function $x(t)$, and now we have two unknown functions $r(t)$ and $\psi(t)$

$$\ddot{x}(t) = -\dot{r}(t) \sin(t + \psi(t)) - r(t) \cos(t + \psi(t)) (1 + \dot{\psi}(t))$$

ODE becomes:

$$\begin{cases} -r \sin(t + \psi) - r \cos(t + \psi)(1 + \dot{\psi}) + r \cos(t + \psi) = \varepsilon f(r \cos(t + \psi), -r \sin(t + \psi)) \\ \rightarrow \dot{r} \sin(t + \psi) + r \dot{\psi} \cos(t + \psi) = -\varepsilon f(r \cos(t + \psi), -r \sin(t + \psi)) \end{cases}$$

Lagrange assumed that:

$$\begin{cases} \dot{x} = \underbrace{\dot{r} \cos(t + \psi)}_{\dot{r} \cos(t + \psi)} - r \sin(t + \psi) (1 + \dot{\psi}) \\ \rightarrow \dot{r} \cos(t + \psi) - r \dot{\psi} \sin(t + \psi) = 0 \end{cases} \quad \dot{r} \cos(t + \psi)$$

$$\begin{cases} \dot{r} \sin(t + \psi) + r \dot{\psi} \cos(t + \psi) = -\varepsilon f(\dots, \dots) \\ \dot{r} \cos(t + \psi) - r \dot{\psi} \sin(t + \psi) = 0 \end{cases}$$

$$\begin{cases} \dot{r} = -\varepsilon \sin(t + \psi) f(r \cos(t + \psi), -r \sin(t + \psi)) \\ \dot{r} \dot{\psi} = -\varepsilon \cos(t + \psi) f(r \cos(t + \psi), -r \sin(t + \psi)) \end{cases}$$

$\cdot \sin(t + \psi)$
 $\cdot \cos(t + \psi)$
 $\cdot \cos(t + \psi)$
 $\cdot -\sin(t + \psi)$

So, $r(t)$ and $\psi(t)$ are slowly varying in t because of ε in RHS
 → assume r, ψ = constant for very small ε

RHSs can be written as Fourier series in fast time t

$$g_0(r, \psi) + \sin t \cdot h_1(r, \psi) + \cos t \cdot g_1(r, \psi) + \sin(2t) \cdot h_2(r, \psi) + \cos(2t) \cdot g_2(r, \psi) + \dots$$

Averaging over the fast time t from 0 to 2π leads to: $\frac{1}{2\pi} \int_0^{2\pi} \dots dt$

$$\frac{1}{2\pi} \cdot g_0(r, \psi) \cdot 2\pi$$

Example $\ddot{x} + x = \varepsilon(-\dot{x} + x^2)$

$$\begin{cases} \dot{x}(t) = r(t) \cos(t + \psi(t)) \\ \dot{y}(t) = r(t) \sin(t + \psi(t)) \end{cases}$$

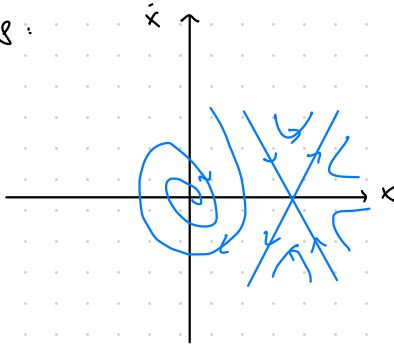
$$\begin{cases} \dot{r} = -\varepsilon \sin(t + \psi) (r \sin(t + \psi) + r^2 \cos^2(t + \psi)) \\ r\dot{\psi} = -\varepsilon \cos(t + \psi) (r \sin(t + \psi) + r^2 \cos^2(t + \psi)) \end{cases}$$

Averaging over the fast variable t again $r, \psi \approx \text{constant}$ because they start with ε

$$\begin{cases} \dot{r}_a = -\varepsilon \cdot \frac{1}{2} r_a \\ \dot{\psi}_a = 0 \end{cases} \Rightarrow \begin{cases} r_a(t) = r_a(0) e^{-\frac{1}{2}\varepsilon t} \\ \psi_a(t) = \psi_a(0) \end{cases}$$

Proof in §11.3: $|x(t) - r_a(t) \cos(t + \psi_a(t))| = O(\varepsilon)$ for $t = O(\frac{1}{\varepsilon})$

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§11.2: Lagrange Standard Form

$$\dot{x} = A(t)x + \varepsilon f(x, \dot{x})$$

$$\underline{\varepsilon = 0}: x(t) = \Phi(t) y$$

$\downarrow$ constant vector

$$\underline{\varepsilon \neq 0}: x(t) = \Phi(t) \cdot y(t) \quad \text{variation of constants}$$

$$\dot{x}(t) = \underbrace{\Phi(t)}_{\Phi_A(t)\Phi(t)} y + \Phi(t) \dot{y} = A\Phi y + \varepsilon f(\Phi y, \dots)$$

$$\hookrightarrow \Phi \dot{y} = \varepsilon f(\Phi y, \dots)$$

$$\hookrightarrow \dot{y} = \varepsilon \Phi^{-1}(t) f(\Phi y, \dots)$$

y is slowly varying in t

§11.3: Averaging in the Periodic Case

$$\dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x; \varepsilon)$$

$$f^0(y) = \frac{1}{T} \int_0^T f(t, y) dt$$

$\downarrow$ keep t constant as slowly varying

$$\dot{y} = \varepsilon f^0(y)$$

(1)

Proof

→ always bounded

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 $T_{\max}$ of f on T

(normalization)

$$\dot{x} = \dot{z} + \varepsilon \frac{\partial u(t, z)}{\partial t} + \varepsilon \frac{\partial u}{\partial z} \dot{z} = \varepsilon f(t, z + \varepsilon u) + \varepsilon^2 g(t, z + \varepsilon u; \varepsilon)$$

$$\hookrightarrow f(t, z) - f^0(z)$$

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Proof: $(1 + \varepsilon \frac{\partial u}{\partial z})(1 + \varepsilon \frac{\partial u}{\partial z})^{-1} = (1 + \varepsilon \frac{\partial u}{\partial z})(1 - \varepsilon \frac{\partial u}{\partial z}) + O(\varepsilon^2)$

$$= I + O(\varepsilon^2)$$

Γ function

In chapter 9: Poincaré-expansion theorem: $\tau = \varepsilon t \Rightarrow \frac{d}{dt} = \varepsilon \frac{d}{d\tau}$

$$\frac{dz}{d\tau} = f^0(z) + O(\varepsilon); \quad \frac{dy}{d\tau} = f^0(y)$$

Theorem tells us: $z(\tau) - y(\tau) = O(\varepsilon)$ for $\tau = O(1)$

$$L_1 \quad z(t) - y(t) = O(\varepsilon) \text{ for } t = O(\frac{1}{\varepsilon})$$

$$\hookrightarrow x(t) - y(t) = z(t) + \varepsilon u(t, z) - y(t) = O(\varepsilon) \text{ for } t = O(\frac{1}{\varepsilon})$$

Example  ; more than one frequency $\rightarrow$ multiple periods

Theorem $\dot{x} = \varepsilon f(t, x) + \varepsilon^2 g(t, x; \varepsilon)$

$$f(t, x) = \sum_{i=1}^2 \underline{f}_i(t, x)$$

T_{T_i} - periodic

Then, $x(t) - y(t) = O(\varepsilon)$ for $t = O(\frac{1}{\varepsilon})$

Theorem $f^0(x) = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau f(t, x) dt$ should exist

11.3

$$\dot{y} = \varepsilon f^0(y)$$

$$x(t) - y(t) = O(\delta(\varepsilon)) \text{ for } t = O\left(\frac{1}{\varepsilon}\right)$$

$$\delta(\varepsilon) = \sup \sup_{\varepsilon} \left\| \int_0^t (f(s, x) - f^0(x)) ds \right\|; x \in D; 0 \leq \varepsilon t \leq c$$

§11.5 Adiabatic invariants (or first integrals)

$$\dot{x} = \varepsilon f(t, \varepsilon t, x)$$

$$\ddot{x} + \omega^2(\varepsilon t)x = 0$$

$$\tau = \varepsilon t \Rightarrow \dot{\tau} = \varepsilon$$

$$\begin{cases} \dot{x} = \varepsilon f(t, \tau, x) \\ \dot{\tau} = \varepsilon \end{cases} \Rightarrow \begin{pmatrix} x \\ \tau \end{pmatrix}' = \varepsilon \begin{pmatrix} f(t, \tau, x) \\ 1 \end{pmatrix} = \bar{X} = \varepsilon \bar{F}(t, \bar{X})$$

Example $\ddot{x} + \omega^2(\varepsilon t)x = 0$

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Example $\ddot{x} + \omega^2(\varepsilon t) x = 0$

$$\tau = \varepsilon t; \quad 0 < a < \omega(\tau) < b$$

$$\left| \frac{d\omega(\tau)}{d\tau} \right| < c$$

$$\dot{x} = \omega(\varepsilon t) y \Rightarrow \ddot{x} = \frac{d\omega(\varepsilon t)}{d(\varepsilon t)} \frac{d(\varepsilon t)}{dt} \cdot y + \omega(\varepsilon t) \dot{y} = -\omega^2(\varepsilon t) x$$

$$\begin{cases} \dot{x} = \omega(\varepsilon t) y \\ \dot{y} = -\frac{\varepsilon}{\omega} \frac{d\omega(\tau)}{d\tau} y - \omega(\varepsilon t) x \end{cases}$$

$$\begin{cases} x = r \sin \phi \\ y = r \cos \phi \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{r} \sin \phi + r \cos \phi \dot{\phi} = \omega(\tau) r \cos \phi \\ \dot{y} = \dot{r} \cos \phi - r \sin \phi \dot{\phi} = -\omega(\tau) r \sin \phi \end{cases}$$

$\cdot \cos \phi$
 $\cdot \sin \phi$
 $\cdot -\sin \phi$
 $\cdot \cos \phi$

$$\begin{aligned} &= \begin{cases} \dot{r} = -\frac{\varepsilon}{\omega} \frac{d\omega(\tau)}{d\tau} r \cos^2 \phi \\ r \dot{\phi} = \omega(\tau) r + \frac{\varepsilon}{\omega} \frac{d\omega(\tau)}{d\tau} r \cos \phi \sin \phi \end{cases} \\ &= \end{aligned}$$

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$$\begin{cases} \dot{r} = -\frac{\varepsilon}{\omega} \frac{d\omega(\tau)}{d\tau} r \cos^2 \phi \\ \dot{\phi} = \omega(\tau) + \frac{\varepsilon}{\omega} \frac{d\omega(\tau)}{d\tau} \cos \phi \sin \phi \\ \dot{\tau} = \varepsilon \end{cases}$$

$\dot{\phi} \approx \omega(\tau)$ which is bounded
 $\hookrightarrow \phi$ incr. as t incr.

If t increases, so does $\phi \Rightarrow \phi$ is a time like variable

$$\begin{aligned} \frac{dr}{d\phi} &= \frac{-\frac{\varepsilon}{\omega} \frac{d\omega(\tau)}{d\tau} r \cos^2 \phi}{\omega(\tau) + \frac{\varepsilon}{\omega} \frac{d\omega(\tau)}{d\tau} \cos \phi \sin \phi} \\ \frac{d\tau}{d\phi} &= \frac{\varepsilon}{\omega(\tau) + \frac{\varepsilon}{\omega} \frac{d\omega(\tau)}{d\tau} \cos \phi \sin \phi} \end{aligned}$$

Integrals of this to average are hard. Since ε is small we can do Taylor expansion of denominator

$$\left[\frac{1}{1 \pm \varepsilon} = 1 - \varepsilon + O(\varepsilon^2) \right]$$

$$\begin{cases} \frac{dr}{d\varphi} = -\frac{\varepsilon}{\omega} \frac{d\omega}{d\tau} r \cos^2 \varphi \cdot \frac{1}{\omega} \left(1 - \frac{\varepsilon}{\omega^2} \frac{d\omega}{d\tau} \cos \varphi \sin \varphi + O(\varepsilon^2) \right) \\ \frac{d\tau}{d\varphi} = \frac{\varepsilon}{\omega} \left(1 - \frac{\varepsilon}{\omega^2} \frac{d\omega}{d\tau} \cos \varphi \sin \varphi + O(\varepsilon^2) \right) \end{cases}$$

Average up to $O(\varepsilon)$:

$$\begin{cases} \frac{dr_a}{d\varphi} = -\frac{\varepsilon}{\omega(\tau_a)} \cdot \frac{d\omega(\tau_a)}{d\tau_a} r_a \cdot \frac{1}{2} \\ \frac{d\tau_a}{d\varphi} = \frac{\varepsilon}{\omega(\tau_a)} \end{cases}$$

If we know $\omega(\tau)$ we can solve. Also:

$$\frac{dr_a}{d\tau_a} = -\frac{1}{\omega(\tau_a)} \frac{d\omega(\tau_a)}{d\tau_a} \frac{r_a}{2} \rightarrow \text{separation of variables:}$$

$$\frac{1}{r_a} \frac{dr_a}{d\tau_a} = -\frac{1}{2\omega(\tau_a)} \frac{d\omega(\tau_a)}{d\tau_a}$$

$$\ln(r_a(\tau_a)) = -\frac{1}{2} \ln(\omega(\tau_a)) + \text{constant}$$

$$\omega(\tau_a)^{1/2} \cdot r_a(\tau_a) = \text{constant} + O(\varepsilon)$$

§11.6 Averaging over one angle; resonance manifolds

$$\begin{cases} \dot{x} = \varepsilon \tilde{X}(\varphi, x) + O(\varepsilon^2) \\ \varphi = Q(x) \end{cases}$$

$\tilde{X} \varphi \in S^1 = T$ -periodic

Q Bounded away from 0

$$\frac{dx}{d\varphi} = \frac{\varepsilon \tilde{X}(\varphi, x)}{Q(x)} + O(\varepsilon^2)$$

$$\int_{\varphi \in S^1} \frac{\tilde{X}(\varphi, x)}{Q(x)} d\varphi = \frac{dy}{d\varphi} = \varepsilon \int_{\varphi \in S^1} \frac{\tilde{X}(\varphi, y)}{Q(x)} d\varphi$$

$$x(\varepsilon) - y(\varepsilon) = O(\varepsilon) \quad \text{for } t = O\left(\frac{1}{\varepsilon}\right)$$

Example

$$\begin{cases} \dot{x} = \frac{1}{2}\varepsilon - \varepsilon \cos \varphi \\ \dot{\varphi} = x - 1 \end{cases} \rightarrow Q \text{ not bounded away from 0}$$

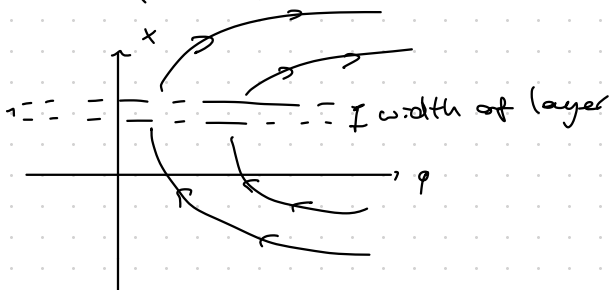


Outside neighborhood of $x=1$:
 φ is time line

Outside the region we can average:

$$\frac{dx}{d\varphi} = \frac{\frac{1}{2}\varepsilon - \varepsilon \cos \varphi}{x-1} \Rightarrow \frac{dx}{d\varphi} = \frac{\frac{1}{2}\varepsilon}{x-1}$$

$$\frac{1}{2}\varepsilon x^2 - x = \frac{1}{2}\varepsilon \varphi + \text{constant}$$



Let $x = 1 + \delta(\varepsilon)\xi$:
unknown order function

$$\begin{cases} \dot{x} = \delta(\varepsilon)\dot{\xi} = \varepsilon(\frac{1}{2} - \cos \varphi) \\ \dot{\varphi} = \delta(\varepsilon)\dot{\xi} \end{cases} \Rightarrow \begin{cases} \dot{\xi} = \frac{\varepsilon}{\delta(\varepsilon)}(\frac{1}{2} - \cos \varphi) \\ \dot{\varphi} = \delta(\varepsilon)\dot{\xi} \end{cases}$$

We can no longer average as soon as the variation in x, ξ are equal to the variation in φ :

$$\frac{\varepsilon}{\delta(\varepsilon)} = \delta(\varepsilon) \Rightarrow \delta(\varepsilon) = \sqrt{\varepsilon}$$

width of layer : $O(\sqrt{\varepsilon})$

$$\begin{cases} \dot{\xi} = (\frac{1}{2} - \cos \varphi)\sqrt{\varepsilon} \\ \dot{\varphi} = \xi\sqrt{\varepsilon} \end{cases} \Rightarrow \text{eq. pts: } \begin{cases} \cos \varphi = \frac{1}{2} \\ \xi = 0 \Rightarrow x = 1 \end{cases}$$

width saddle points and center points

§11.7 Averaging over more than 2 angle

$$\begin{cases} \dot{x} = \varepsilon X(\varphi_1, \varphi_2, x), \quad x \in \mathbb{R} \\ \dot{\varphi}_1 = x \\ \dot{\varphi}_2 = 1 \end{cases}$$

$$X(\varphi_1, \varphi_2, x) = \sum_{k_1, k_2 = -\infty}^{\infty} c_{k_1, k_2} e^{i(k_1 \varphi_1 + k_2 \varphi_2)} \quad \rightarrow \varphi_1, \varphi_2 \text{ periodic}$$

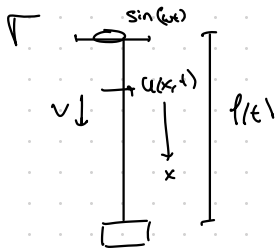
$$X(\varphi_1, \varphi_2, x) = c_{1,1} \omega \left(\sin(\varphi_1 - \varphi_2) - 3 \sin(\varphi_1 + \varphi_2) \right)$$

$\dot{\varphi}_1 + \dot{\varphi}_2 \leq 0 \Rightarrow x + 1 \approx 0$
 $\hookrightarrow$ varying slowly if $\varphi_1 - \varphi_2 \approx 0 \Rightarrow x - 1 \approx 0$

In general, we can not average if $k\phi_1 + l\phi_2 \approx 0$: $kx + l \approx 0$
 $\hookrightarrow x = -\frac{l}{k}$

Example $\ddot{x} + x = \varepsilon \sin(f(t)) = \varepsilon \sin(f(t_0) + \underbrace{f'(t_0)(t-t_0)}_{\text{Resonance when } f'(t_0) \approx \pm 1} + \text{h.o.t.})$

Take $f(t) = \varepsilon t^2$



$$u(0,t) = \varepsilon \sin(\omega, t)$$

$$u_{tt} - u_{xx}$$

$$u_{tt} + 2\varepsilon u_{xt} + \varepsilon^2 u_{xx} - c^2 u_{xx} = \dots$$

$$\frac{x}{l(t)} = \xi \rightarrow 0 \leq \xi \leq 1$$

$$\hookrightarrow u_{tt} \dots - \frac{c^2}{l^2(t)} u_{\xi\xi} = \dots \quad \text{time-dep. coefficients}$$

Ham. BC $\Rightarrow$ PDE contains $\sin(\omega t)$

$$u(\xi, t) = \sum a_n(t) \sin(n\pi \xi)$$

$$u_n''(t) + \frac{(n\pi)^2 c^2}{l^2(t)} u_n = \dots \sin(\omega t)$$

$$f(t) = l_0 + \varepsilon t$$

$$u_n'' + (n\pi)^2 u_n = \varepsilon \left(\dots + \sin\left(\frac{R l_0}{\varepsilon} (\varepsilon t^2 - 1)\right) + \dots \right)$$

$$t^+ = \int \frac{1}{f(s)} ds$$

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \\ f(t) = \varepsilon t^2 \end{cases} \Rightarrow \begin{cases} r' = \varepsilon \sin \varphi \sin f \\ \varphi' = -1 + \varepsilon \sin \varphi \cos \varphi \\ f' = 2\varepsilon t = 2\varepsilon \\ \tau' = \varepsilon \end{cases}$$

$$\begin{cases} r' = \frac{\varepsilon}{2} (\cos(\varphi - f) - \cos(\varphi + f)) \\ \varphi' = -1 + \frac{\varepsilon}{2} (\sin(f + \varphi) - \sin(f - \varphi)) \\ f' = 2\varepsilon \\ \tau' = \varepsilon \end{cases}$$

We can average as long as $\varphi - f$ and $\varphi + f$ are fast varying variables

So not when $\varphi - f \approx 0$ and/or $\varphi + f \approx 0$

$$-1 - 2\tau \rightarrow \text{Newer} \approx 0$$

$$-1 + 2\tau \rightarrow \tau \approx \frac{1}{2}$$

$$\text{Rescale: } \tau = \frac{1}{2} + \delta(\varepsilon)\Theta ; \quad \Phi = f + \rho$$

$$\begin{cases} r' = \frac{\varepsilon}{2} (\cos(\varphi - f) - \cos(\Phi)) \\ \varphi' = -1 + \frac{\varepsilon}{2} (\sin(\Phi) + \sin(f - \varphi)) \\ f' = 1 + 2\delta(\varepsilon)\Theta \\ \Theta' = \frac{\varepsilon}{\delta(\varepsilon)} \\ \Phi' = \delta(\varepsilon)\Theta + \frac{\varepsilon}{2} (\sin(\Phi) + \sin(f - \varphi)) \end{cases}$$

$$\hookrightarrow \delta(\varepsilon) = \sqrt{\varepsilon}$$